Monodromy Criterion for the Good Reduction of Surfaces

Genaro Hernández Mada

A Thesis in the Department of Mathematics and Statistics

Presented in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy (Mathematics) at Concordia University

July 2015

CONCORDIA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

This is to certify that the thesis prepared

By: Genaro Hernandez-Mada

Entitled: Monodromy Criterion for the Good Reduction of Surfaces

and submitted in partial fulfillment of the requirements for the degree of

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

______________________ Chair

Dr. O. Brinon ______________________ External Examiner

______________________ External to Program

Dr. B. Chiarellotto ______________________ Examiner

Dr. F. Andreatta ______________________ Examiner

Dr. A. Iovita ______________________ Thesis Supervisor

Approved by

____
Chair of Department or Graduate Program Director

July 3, 2015

____
Dean of Faculty

Abstract

Monodromy Criterion for the Good Reduction of $K3$ Surfaces.

Genaro Hernández Mada, PhD.

Concordia University, 2015.

Let $p > 3$ be a prime number and K a finite extension of $\mathbb{Q}_p$. We consider a proper and smooth surface X_K over K, with a semistable model X over the ring of integers O_K of K. In this thesis, we give a criterion for the good reduction of X_K for the case of $K3$ surfaces, in terms of the monodromy operator in the second De Rham cohomology group $H^2_{DR}(X_K)$.

We don't use trascendental methods nor p-adic Hodge Theory as in other works concerning this problem (such as [Ma14], [LM14] and [Pe14]). Instead, we first get a p-adic version of the Clemens-Schmid exact sequence and use it to study the degree of nilpotency of the monodromy operator N on the log-crystalline cohomology group of the special fiber X_s of the semistable model X.

By the work of Nakkajima ([Na00]), we can assume that X_s is a combinatorial $K3$ surface. Then, we prove that X_s is of type I iff $N = 0$; X_s if of type II iff $N \neq 0, N^2 = 0$; X_s is of type III iff $N^2 \neq 0$. In particular, this implies that X_K has good reduction if and only if the monodromy operator on $H^2_{DR}(X_K)$ is zero.

Finally, we also give some ideas on how to address the same problem for the case of Enriques surfaces. In particular, we prove that we are reduced to the case of $K3$ surfaces.

Acknowledgment

I would like to thank infinitely my supervisors Bruno Chiarellotto from Padova and Adrian Iovita from Concordia for all the great mathematical knowledge that they allowed me to learn during the development of this project, and for their patience explaining me difficult concepts.

I would also like to thank Olivier Brinon from Bordeaux, who I had very fruitful discussions with, during my period in that university.

The Algant Consortium gave me the financial support and the opportunity to develop this project under great academic conditions. I would like to thank them, and especially to the three universities that were directly involved: Concordia University, University of Padova and University of Bordeaux.

Finally, I thank any person that in some way contributed to the development of this project. This includes my fellow PhD students that I had discussions with, administrative personal, my friends and my family who gave me their support even when we are very far.

Contents

1 Introduction **1**

2 Preliminaries **6**

2.1 Logarithmic Structures 6

2.1.1 Pre-log and Log Structures 6

2.1.2 Log smooth and Log étale morphisms 8

2.2 Log-Deformations 9

2.2.1 Pro-representable functors and hulls 9

2.2.2 Deformation functors 11

2.3 Cohomology Theories 12

2.3.1 Rigid Cohomology 12

2.3.2 Rigid Cohomology with support on a closed subscheme 15

2.3.3 Rigid Cohomology with Compact Support 16

2.3.4 Relative Log Crystalline Cohomology 17

2.4 Néron-Popescu Desingularization 21

3 Generalities on $K3$ surfaces **25**

3.1 $K3$ Surfaces over $\mathbb{C}$ 25

3.2 $K3$ Surfaces over more General Fields 26

3.3 Log $K3$ Surfaces 28

4 Clemens-Schmid exact sequence over a local basis **33**

4.1 Classical Clemens-Schmid exact sequence 34

4.2 Chiarellotto-Tsuzuki's work 38

4.3 Notation and setting over a local basis 40
4.4 A construction using Néron-Popescu desingularization 41
4.5 Relative Cohomology 43
4.6 Reduction to the case of a family over a curve 48

5 Monodromy Criteria 52

5.1 General Monodromy Criteria 52
5.2 The Main Theorem 57

6 The case of Enriques Surfaces 64

6.1 Universal coverings of Enriques and Rational schemes in characteristic p . . . 65
6.2 Enriques Surfaces and Monodromy 66

Bibliography 68

Chapter 1

Introduction

Let p be a prime number and K a finite extension of $\mathbb{Q}_p$. Consider a smooth, proper and geometrically irreducible scheme X_K over Spec K.

Definition 1.0.1. We say that X_K has good reduction if there exists a smooth, proper scheme $\mathfrak{X}$ over O_K with generic fiber X_K, i.e.,

$$X_K \cong \mathfrak{X} \times_{\mathrm{Spec}\ O_K} \mathrm{Spec}\ K$$

The question of whether X_K has good reduction or not can be answered via ℓ-adic or p-adic criteria in some cases. For example, if $X_K = A_K$ is an abelian variety and G_K the absolute Galois group of K, we get that A_K has good reduction if and only if for all $\ell \neq p$ (equivalently, for some $\ell \neq p$), the ℓ-adic G_K-representation $T_\ell(A_K)$ is unramified (see [ST68]). The p-adic criterion says that A_K has good reduction if and only if the p-adic G_K-representation $T_p(A_K)$ is crystalline (see Theorem II.4.7 in [CI99] and Corollaire 1.6 in [Br00]). Recall that in general, for semistable p-adic representations, this is equivalent to having trivial monodromy operator.

For more general varieties, the criteria from the preceding paragraph are not valid, but in some cases, different criteria can be obtained. For example, if X_K is a curve with semistable reduction, Oda (in [Oda95]) obtained an ℓ-adic criterion looking at the Galois action on the étale fundamental group and Andreatta-Iovita-Kim (in [AIK13]) obtained its p-adic version studying the monodromy action on the De Rham fundamental group. This means that it is not enough to look at the first cohomology group with its Galois/monodromy action, but one needs to look at the whole fundamental group (i.e., not only its abelianization).

In this thesis, we shall obtain a p-adic criterion for the analogous situation for $K3$ surfaces. Namely, we suppose that $p > 3$ and X_K is a smooth, projective $K3$ surface over Spec K having a minimal semistable model X over the ring of integers O_K of K. We may assume to have combinatorial reduction (see proposition 3.4 in [Na00]). Then, since we are dealing with $K3$ surfaces, the first De Rham cohomology group is trivial, as well as the connected De Rham fundamental group. Now we look at the monodromy action on the higher De Rham cohomology groups $H^i_{\mathrm{DR}}(X_K)$. The monodromy N is given in the frameweork of the theory of log-schemes and log-crystalline cohomology (see [HK94]). Then our result is the following:

Under the hypotheses above, the $K3$ surface X_K has good reduction if and only if the monodromy N is zero on $H^2_{DR}(X_K)$.

In fact we shall get more than that. We know that the operator N is always nilpotent (N^3 is always trivial). In case of not having good reduction, we can refine the theorem just stated: the type of bad reduction is determined by the order of nilpotency of N. For the complete result, see theorem 5.2.1.

One can also note that once that we obtain this criterion for good reduction in terms of the monodromy operator on log-crystalline cohomology, we also get an étale one. Namely: X_K has good reduction if and only if $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline representation. This is a consequence of our criterion and the comparison theorems in [Ts99].

Over the complex numbers, the analogue of the previous situation can be understood as a semistable family of varieties over the complex unit disk. Given a semistable degeneration of $K3$ surfaces, the works of Kulikov ([Ku77]), Persson-Pinkham ([PP81]) and Morrison ([Mo84]) show how the monodromy action on the generic fiber determines the behavior of the special one. We state the most important results for this classical situation in section 4.1. To prove this, one uses all the information coming from the structure of the family: weight-monodromy conjecture and Clemens-Schmid exact sequence. Our proof has been inspired by these methods.

The monodromy on the De Rham cohomology of X_K is given by the monodromy operator on the log-crystalline cohomology of the special fiber X_s (which is in characteristic p)

endowed with the induced log-structure (see for example [HK94]). Using Nakkajima's results on deformations of $K3$ surfaces ([Na00]) we may construct a log-smooth deformation of our special fiber over the ring of formal power series $k[[t]]$, where k is the residue field of K. Then, using a Popescu's version of Artin approximation, we can get a deformation of X_s over a smooth scheme Y over $k[t]$ (possibly of dimension larger than 1). Finally, by taking a well-chosen curve inside Y, we are reduced to the case of a family over a smooth curve, so we can use Chiarellotto-Tsuzuki's results for this setting. In particular, for such a family, we can use the weight-monodromy conjecture and the existence of a Clemens-Schmid type exact sequence. This gives the elements to rephrase Kulikov-Persson-Pinkham's and Morrison's results in characteristic p, allowing us to get our main theorem (theorem 5.2.1), which is similar to the one obtained by Pérez Buendía in [Pe14]. This method of proof is completely different to the one used by Matsumoto in [Ma14] and by Liedtke and Matsumoto in [LM14], who obtain results related to ours.

One can expect to use our methods to study the case of semistable Enriques surfaces. Indeed, we can follow our techniques along the lines we used for $K3$ surfaces, since again by Nakkajima's work ([Na00]) we have a classification of the possible special fibers. In chapter 6 we describe the ideas to treat this problem.

Let us give an outline of this thesis. We begin in chapter 2 stating the most basic tools that we use in the proof of our main theorem. These are logarithmic structures, log-deformations, cohomology theories and Néron-Popescu desingularization. Then, in chapter 3 we give a brief study of $K3$ surfaces, first over $\mathbb{C}$ and then for more general fields, such as fields in characteristic $p > 0$.

Chapter 4 is devoted to get a generalization of Clemens-Schmid exact sequence in characteristic $p > 0$. First we begin by recalling the classical Clemens-Schmid exact sequence and an application of it. Then, we make a brief description of Chiarellotto and Tsuzuki's work on a p-adic version of the Clemens-Schmid exact sequence. Then, we proceed to work on our situation of study, describing first the geometric situation and then using Néron-Popescu desingularization (see [Sw95]) to write the ring of formal power series $k[[t]]$ as a limit of smooth $k[t]$-algebras:

$$k[[t]] = \varinjlim_{\alpha} A_{\alpha}.$$

This allows, in a similar way to what is done in section 4 of [It05], to see our situation as a fiber inside a larger family of varieties $f : X_A \to Y = \text{Spec } A$, where $A = A_\alpha$ for some α. Then, we can use the relative cohomology theories defined and studied by Shiho in [Sh08], which give relative cohomology sheaves on a formal scheme, that is a smooth lifting $\mathcal{Y}$ of Y. This is useful once that we have a deformation, which we have by the results in chapter 3.

In section 4.5 we state the results on relative cohomology that are useful for our purposes. In particular, we need the base change theorem and the comparison isomorphisms between the different cohomology theories (log-crystalline, log-convergent and log-analytic), since these are needed to use the results in [CT12]. This means that the relative cohomology sheaves, defined on the large family, satisfy the desired properties.

Then, in section 4.6, we construct a smooth curve C inside Y in such a way that we can restrict the family $X_A \to Y$, as well as the cohomology sheaves, to a smaller family X_C over this curve. In particular, this allows us to use the main result in [CT12] and get the version of the Clemens-Schmid exact sequence for this setting:

$$\cdots \to H^m_{rig}(X_s) \to H^m_{log-crys}((X_s, M_s)/W^\times) \otimes K_0 \xrightarrow{N}$$

$$H^m_{log-crys}((X_s, M_s)/W^\times) \otimes K_0(-1) \to H^{m+2}_{X_s,rig}(X_C) \to H^{m+2}_{rig}(X_s) \to \cdots$$

In chapter 5 we get out main result. First in section 5.1 we use the Clemens-Schmid exact sequence in characteristic p to get criteria for N to be the zero map on $H^1_{log-crys}$ or $H^2_{log-crys}$, assuming that we are dealing with a semistable family of varieties over a smooth curve over a finite field. For this, we use the fact that the monodromy and weight filtrations on the special fiber coincide ([CT12]). If we do not assume that the special fiber is inside a semistable family of varieties, this is known only for the case of curves and surfaces (see [Mk93]). The criteria that we get in this section are in terms of the Betti numbers of the dual graph of the special fiber, which can be easily described in the case of combinatorial reduction. As we mentioned before, we can always restrict ourselves to this case.

Finally, in section 5.2, after introducing deformation theory for $K3$ surfaces along the lines of Nakkajima ([Na00]), we apply the criteria from section 5.1 to the case of $K3$ surfaces, assuming that the special fiber is combinatorial, i.e., it is one of three possible types. We obtain that the degree of nilpotency determines the type of degeneracy we will get. Our main result will be stated in theorem 5.2.1, and as a consequence we get that the trivial monodromy action on the second de Rham cohomological group is equivalent to good reduction (corollary 5.2.1).

In the last chapter, we state some ideas for further work. Once that we have found a criterion for good reduction of $K3$ surfaces, we shall treat the analogue problem for Enriques surfaces. As in the case of the criterion for curves (in [AIK13]), an idea is to study some kind of "homotopy" or rational homotopy theory and the monodromy action on it (see for example [Lz14]). Heuristically, an Enriques surface X is not simply connected, but has the same second homotopy group as its universal covering, which is a $K3$ surface X', which is simply connected, hence this second homotopy group is in fact the second cohomology group of X'. Following this idea, in section 6.1 we give a description of the universal covering of an Enriques surface, and then in section 6.2 we state how to apply it in order to get the desired result.

Chapter 2

Preliminaries

In this chapter we recall the basic notions and the results that will be useful for the development of the next chapters. In particular, we present log-structures following [Ka89], log-deformations following [Sc68], [Ka], [Na00], and $K3$ surfaces.

2.1 Logarithmic Structures

The notion of logarithmic structure was formulated by J.M. Fontaine and L. Illusie and developed by K. Kato in [Ka89]. It allows to study a new range of smooth morphisms, which are not necessarily smooth in the classical sense.

2.1.1 Pre-log and Log Structures

We shall consider only commutative monoids with a unit element.

Definition 2.1.1. Let X be a scheme. A pre-log structure on X is a sheaf of monoids M on the étale site $X_{\text{ét}}$ endowed with a morphism $\alpha : M \to \mathcal{O}_X$, where $\mathcal{O}_X$ is considered as a monoid with the operation of multiplication.

Definition 2.1.2. A morphism $(X, M) \to (Y, M)$ of schemes with pre-log structures is a pair (f, h) of a morphism of schemes $f : X \to Y$ and a homomorphism $h : f^{-1}N \to M$ such that the diagram

$$\begin{tikzcd} f^{-1}N \arrow[r, "h"] \arrow[d] & M \arrow[d] \\ f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

is commutative.

Definition 2.1.3. A pre-log structure (M, α) on X is called a logarithmic structure (or log structure) if α induces an isomorphism $\alpha^{-1}\mathcal{O}_X^\times \cong \mathcal{O}_X^\times$. The couple (X, M) is called a log scheme.

Remark 2.1.1. We can make analogue definitions for a formal scheme instead of the scheme X.

An immediate example of log structure on a scheme X is given by the monoid $\mathcal{O}_X^\times$ with the inclusion $\mathcal{O}_X^\times \subset \mathcal{O}_X$. The main example of log structure that we shall use is the following: Let X be a regular scheme and D a reduced divisor with normal crossings. Then, we define M as

$$M = \{g \in \mathcal{O}_X : g \text{ is invertible outside } D\} \subset \mathcal{O}_X.$$

In fact, this is an example of a *fine* log structure (see section 2 of [Ka89]).

Remark 2.1.2. Given a pre-log structure M on X, we can define an associated log structure M^a as the push-out of the diagram

$$\begin{tikzcd} \alpha^{-1}\mathcal{O}_X^\times \arrow[r] \arrow[d] & M \\ \mathcal{O}_X^\times & \end{tikzcd}$$

in the category of sheaves of monoids on $X_{\text{ét}}$, endowed with the morphism $M^a \to \mathcal{O}_X$ defined by the formula

$$(a, b) \mapsto \alpha(a)b \qquad \forall a \in M, b \in \mathcal{O}_X^\times$$

Given a morphism of schemes $f : X \to Y$, and a log structure on Y, the sheaf of monoids $f^{-1}M$ gives a pre-log structure on X via the composition

$$f^{-1}M \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

We define the *inverse image* of M to be the log structure associated to $f^{-1}M$ and we denote it by f^*M. In fact one can also define direct images, but we are not going to need them. The reader can find the definition in page 193 of [Ka89].

2.1.2 Log smooth and Log étale morphisms

As mentioned at the beginning of the chapter, the notion of log schemes allows to define a notion of "smoothness" for morphisms that are not necessarily smooth in the classical sense. In general, a morphism of log schemes $f : (X, M) \to (Y, N)$ is said to be a *closed immersion* if $X \to Y$ is a closed immersion and $f^*N \to M$ is surjective. With this notion, we can define log smoothness and log étaleness, in an analogous way to the usual notions of smoothness and étaleness.

Definition 2.1.4. Let $f : (X, M) \to (Y, N)$ a morphism of log schemes (with M and N fine log structures). We say that f is log smooth (resp. log étale) if for any commutative diagram

$$\begin{tikzcd}
(T', L') \arrow[r] \arrow[d, "i"'] & (X, M) \arrow[d, "f"] \\
(T, L) \arrow[r] & (Y, N)
\end{tikzcd}$$

with L, L' fine log structures, i a closed immersion defined by an ideal I on T with $I^2 = 0$, there exists étale locally on T (resp. there exists a unique) $g : (T, L) \to (X, M)$ making the complete diagram commutative.

In the next chapter we shall see examples of log smooth morphisms. In fact, we shall get an important example by the following criterion:

Proposition 2.1.1. *Let $f : (X, M) \to (Y, N)$ be a morphism of log schemes. Assume we are given a chart $Q_Y \to N$ of N. Then, f is smooth (resp. étale) if and only if étale locally on X, there exists a chart $(P_X \to M, Q_Y \to N, Q \to P)$ of f extending the given $Q_Y \to N$ such that the following two conditions are satisfied:*

(i) The kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups of orders invertible on X.

(ii) The induced morphism $X \to Y \times_{Spec\ \mathbb{Z}[Q]} Spec\ \mathbb{Z}[P]$ is étale (in the classical sense).

The proof can be found in [Ka89] (see theorem 3.5).

2.2 Log-Deformations

In this section we study the other main tool that plays a central role in the development of the theory, specially for what is done in chapter 3. This is the theory of log deformations, developed mainly by F. Kato ([Ka96]), and which is a log version of the theory of Schlessinger ([Sc68]).

2.2.1 Pro-representable functors and hulls

Here we follow [Sc68]. We consider a complete noetherian local ring Λ, with maximal ideal μ and residue field k. Denote by $\mathcal{C}$ the category of artinian local Λ-algebras having residue field k (i.e., the structure morphism $\Lambda \to A$ induces a trivial extension of residue fields). Denote by $\hat{\mathcal{C}}$ the category of complete artinian local Λ-algebras such that $A/\mathfrak{m}^n$ is in $\mathcal{C}$ for all n.

Given a functor $F : \mathcal{C} \to \mathbf{Sets}$ (covariant and such that $F(k)$ has only one element), we extend it to $\hat{\mathcal{C}}$ by

$$\hat{F}(A) := \varprojlim F(A/\mathfrak{m}^n).$$

For any R in $\hat{\mathcal{C}}$, we denote by h_R the functor on $\mathcal{C}$ $h_R(-) = \mathrm{Hom}(R, -)$. Then, for any functor on $\mathcal{C}$ we have a canonical isomorphism

$$\hat{F}(R) \xrightarrow{\sim} \mathrm{Hom}(h_R, F).$$

For the definition of this isomorphism, look at [Sc68]. Take into consideration that the factorization for some n is due to the fact that we are dealing with artinian algebras.

Given a functor $F : \mathcal{C} \to \textbf{Sets}$ and a pro-couple (R, ξ) (this is, $R \in \mathcal{C}$ and $\xi \in \hat{F}(R)$), we say that (R, ξ) *pro-represents* F if the morphism $h_R \to F$ corresponding to ξ is an isomorphism.

Definition 2.2.1. A morphism of functors $F \to G$ is smooth if for any surjection $B \to A$ in $\mathcal{C}$, the morphism

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective.

Remark 2.2.1. If $F \to G$ is smooth, then for any A in $\hat{\mathcal{C}}$, we have $\hat{F}(A) \to \hat{G}(A)$ is surjective.

For any functor $F : \mathcal{C} \to \textbf{Sets}$, we denote by t_F its tangent space, which is defined to be $F(k[\epsilon])$, where $k[\epsilon]$ is the ring of dual numbers. Usually, t_F has an intrinsic structure of vector space (Lemma 2.10 in [Sc68]).

Definition 2.2.2. A pro-couple (R, ξ) is a pro-representable hull of F, or just a hull of F, if the morphism $h_R \to F$ induced by ξ is smooth and the induced map $t_R \to t_F$ is a bijection.

Remark 2.2.2. If the pro-couple (R, ξ) pro-represents F, then it is also a hull. Moreover, in this case (R, ξ) is unique up to canonical isomorphism. In general we only have non canonical isomorphism (for hulls).

Definition 2.2.3. A surjection $p : B \to A$ in $\mathcal{C}$ is called a small extension if $\ker p$ is a nonzero principal ideal annihilated by the maximal ideal of B.

Now we can state the following theorem. For the proof, see [Sc68].

Theorem 2.2.1. *Let F be a functor $\mathcal{C} \to$ **Sets**, such that $F(k)$ contains only one point. Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map*

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{2.1}$$

Then,

(i) *F has a hull if and only if F has the following three properties:*

(a) *(2.1) is a surjection whenever $A'' \to A$ is a small extension.*

(b) *(2.1) is a bijection when $A = k$, $A'' = k[\epsilon]$.*

(c) $\dim_k(t_F) < \infty$.

(ii) *F is pro-representable if and only if F satisfies the three preceding properties and*

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$$

is a bijection for any small extension $A' \to A$

We shall see that theorem 2.2.1 can be used to prove that the functors that we are interested in are pro-representable or at least have a hull. We introduce the first one in the next section.

2.2.2 Deformation functors

Let X be a fixed scheme over k and $A \in \mathcal{C}$. A deformation of X/k to A is a flat scheme Y over A, together with a morphism $i : X \to Y$, such that $X = Y \otimes_A k$. If Y' is another deformation to A, we say that Y and Y' are *isomorphic deformations* if there exists a morphism $f : Y \to Y'$ over A which induces the identity on the closed fiber X. This implies in particular that f is an isomorphism of schemes.

We define $\mathbb{D}(A)$ to be the set of isomorphism classes of deformations of X/k to A. Notice that $\mathbb{D}$ defines a functor $\mathcal{C} \to$ **Sets**. In general, this is not pro-representable, but using theorem 2.2.1, we can prove the following (see [Sc68]):

Proposition 2.2.1. *If X is proper, then $\mathbb{D}$ has a hull (R, ξ). Moreover, (R, ξ) pro-represents $\mathbb{D}$ if and only if for each small extension $A' \to A$, and each deformation Y' of X/k to A', every automorphism of the deformation $Y' \otimes_{A'} A$ is induced by an automorphism of Y'.*

Remark 2.2.3. A special case in which the functor $\mathbb{D}$ is pro-representable, which is treated in [De81], is when k is an algebraically closed field of characteristic $p > 0$, $W = W(k)$ the ring of Witt vectors of k and X is a $K3$ surface over k.

Now we would like to get similar results in the log setting. First of all, let us consider a finitely generated integral saturated monoid Q having no invertible element other than 1. Denote by $\mathcal{C}_{\Lambda[[Q]]}$ the category of artinian local $\Lambda[[Q]]$-algebras with the residue field k, and $\hat{\mathcal{C}}_{\Lambda[[Q]]}$ the category of pro-objects, as in [Sc68].

Definition 2.2.4. Let A be an object of $\mathcal{C}_{\Lambda[[Q]]}$. A log smooth lifting of $f : (X, M) \to (\text{Spec } k, Q)$ on A is a morphism $\widetilde{f} : (\widetilde{X}, \widetilde{M}) \to (\text{Spec } A, Q)$ together with a cartesian diagram

$$\begin{tikzcd} (X, M) \arrow[r] \arrow[d, "f"'] & (\widetilde{X}, \widetilde{M}) \arrow[d, "\widetilde{f}"] \\ (\text{Spec } k, Q) \arrow[r] & (\text{Spec } A, Q) \end{tikzcd}$$

With this definition, we define a functor $\mathbb{D}_{(X,M)} : \mathcal{C}_{\Lambda[[Q]]} \to \textbf{Sets}$ by associating to each A the set of isomorphism classes of log smooth liftings of f on A.

Remark 2.2.4. We use an abuse of notation by writing $\mathbb{D}_{(X,M)}$, and not making emphasis on the map f and/or the log structure on the base log point, but for our purposes, this shouldn't cause confusion.

The main result in [Ka96], which is proven using the criterion that we gave in the preceding section, is the following:

Theorem 2.2.2. *If X is proper over k, then $\mathbb{D}_{(X,M)}$ has a hull.*

Remark 2.2.5. As in the smooth case, the case of $K3$ surfaces is of special interest. In particular, in that case $\mathbb{D}_{(X,M)}$ is pro-representable. In [Na00], Nakkajima uses this functor and the log enlarged formal Brauer group (see [AM77]) to construct a lifting of semistable $K3$ surfaces which will be useful in the following chapters. We treat this in chapter 3. Moreover, Nakkajima also considers the functor of deformations not only of the log-scheme but the log-scheme with a line bundle L. That one is pro-representable as well.

2.3 Cohomology Theories

2.3.1 Rigid Cohomology

In this section we state the definitions and results that are useful for our work. For the proofs, see [Be97ii]. A more complete treatment of Rigid Cohomology can be found at [LS]. We assume the reader familiar with rigid analytic geometry, as in [BGR] or [FV].

We fix a field k of characteristic $p > 0$, and a complete discrete valuation ring $\mathcal{V}$ with residue field k. We denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{V}$ and by K its fraction field. We suppose moreover that the valuation v is such that $v(p) = 1$ and we use the absolute value given by $|p| = p^{-1}$.

Let $\mathcal{P}$ be a p-adic formal scheme locally topologically of finite presentation over $\mathcal{V}$. Recall that there is an algebraic k-variety $\mathcal{P}_k$, called *special fiber of* $\mathcal{P}$, which is homeomorphic as topological space to $\mathcal{P}$. This allows to identify $\mathcal{P}$ and $\mathcal{P}_k$ as topological spaces. There is also a quasi-separated rigid analytic variety over K, denoted by $\mathcal{P}_K$ and called *generic fiber* of $\mathcal{P}$. This is equipped with a map of ringed spaces

$$sp : \mathcal{P}_K \to \mathcal{P},$$

called *specialization map*.

Example 2.3.1. Suppose that A is a complete p-adic $\mathcal{V}$-algebra topologically of finite presentation. If $\mathcal{P} = \mathrm{Spf}\ A$, then

$$\mathcal{P}_k = \mathrm{Spec}\ (A \otimes_{\mathcal{V}} k), \quad \mathcal{P}_K = \mathrm{Spm}(A \otimes_{\mathcal{V}} K).$$

The specialization map $sp : \mathcal{P}_K \to \mathcal{P}$ is given by

$$x \mapsto \ker(A \to \mathcal{V}(x) \to K(x)),$$

where $K(x)$ denotes the residue field of $x \in \mathcal{P}_K$ and $\mathcal{V}(x)$ its valuation ring.

For any closed subset X of $\mathcal{P}_k$, we define the *tube of* X *in* $\mathcal{P}$ as

$$]X[_{\mathcal{P}} := sp^{-1}(X).$$

It is an open subset of $\mathcal{P}_K$, hence it has a structure of rigid analytic space.

Now suppose that $\overline{X}$ is a closed subschema of $\mathcal{P}_k$ with an open immersion $j : X \to \overline{X}$. Let $Z = \overline{X} - X$. An open $V \subset]\overline{X}[_{\mathcal{P}}$ is said to be a *strict neighborhood of* $]X[_{\mathcal{P}}$ *in* $]\overline{X}[_{\mathcal{P}}$ if $\{V,\ ]Z[_{\mathcal{P}}\}$ is an admissible covering of $]\overline{X}[_{\mathcal{P}}$. For any strict neighborhood V, denote by j_V the inclusion

$$V \hookrightarrow]\overline{X}[_{\mathcal{P}}.$$

For any abelian sheaf $\mathcal{F}$ on $]\overline{X}[_{\mathcal{P}}$, we define

$$j^{\dagger}\mathcal{F} := \varinjlim_{V} j_{V*}j_V^{-1}\mathcal{F},$$

where the limit is taken over all strict neighborhoods of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$. This defines an exact functor from the category of abelian sheaves on $]\overline{X}[_{\mathcal{P}}$ to itself. We define another exact functor $\Gamma^{\dagger}_{]Z[}$ via the short exact sequence

$$0 \to \Gamma^{\dagger}_{]Z[}\mathcal{F} \to \mathcal{F} \to j^{\dagger}\mathcal{F} \to 0.$$

With the definitions that we have given in this section, we can define the rigid cohomology spaces. Let X a separated scheme over k and take a Nagata compactification $j : X \hookrightarrow \overline{X}$ over k. Suppose that there exists a closed immersion into a smooth formal $\mathcal{V}$-scheme

$$\overline{X} \hookrightarrow \mathcal{P}.$$

Then, we define the *rigid cohomology* of X as the K-vector space

$$H^*_{rig}(X) := H^*(]\overline{X}[_{\mathcal{P}}, j^{\dagger}\Omega^{\bullet}_{]\overline{X}[_{\mathcal{P}}}).$$

Remark 2.3.1. In general, we cannot find such an immersion into a smooth formal $\mathcal{V}$-scheme. This can be done only locally, and then one defines the rigid cohomology via a Čech complex. More details can be found in section 2 of [Ch99].

Rigid cohomology is compatible with extensions of scalars:

Proposition 2.3.1. *Let K' be an extension of K, with ring of integers $\mathcal{V}'$ and residue field*

k', X a separated k-scheme of finite type and $X' = X \otimes_k k'$. Then, there is a canonical isomorphism

$$K' \otimes_K H^*_{rig}(X) \xrightarrow{\sim} H^*_{rig}(X')$$

The final two propositions of this section indicate that rigid cohomology generalizes both crystalline and Monsky-Washnitzer cohomology.

Proposition 2.3.2. *Let X be a proper and smooth over k. If W is a Cohen ring of k, there is a canonical isomorphism*

$$H^*_{rig}(X) \xrightarrow{\sim} H^*_{crys}(X/W) \otimes_W K.$$

Proposition 2.3.3. *Let X be affine and smooth over k. Denote by H^*_{MW} its Monsky-Washnitzer cohomology. Then, there is a canonical isomorphism*

$$H^*_{rig}(X) \xrightarrow{\sim} H^*_{MW}(X).$$

2.3.2 Rigid Cohomology with support on a closed subscheme

In this section we define rigid cohomology with support on a closed subscheme. Again our main reference for the proofs of all the results is [Be97ii].

Let X be a k-scheme of finite type and $Z \subset X$ a closed subscheme. Take $j_X : X \hookrightarrow \overline{X}$ a compactification of X over k, and $\overline{Z} \subset \overline{X}$ a closed subscheme such that $\overline{Z} \cap X = Z$. We define the *rigid cohomology of X with support on Z* by

$$H^*_{Z,rig}(X) := H^*(]\overline{X}[_{\mathcal{P}}, \Gamma^\dagger_{]Z[} j^\dagger_X \Omega^\bullet_{]\overline{X}[_{\mathcal{P}}}).$$

If X is smooth, then they are finite dimensional K-vector spaces.

Remark 2.3.2. These cohomology spaces do not depend (up to canonical isomorphism) on the choice of $\overline{Z}, \overline{X}, \mathcal{P}$.

Remark 2.3.3. For a fixed X, these cohomology spaces depend only on the topological space of Z, and not on its subscheme structure. Moreover, if $Z = X$, then we can take $\overline{Z} = \overline{X}$, and consequently

$$H^*_{X,rig}(X) = H^*_{rig}(X).$$

Rigid cohomology and rigid cohomology with support on a closed subscheme satisfy the usual long exact sequence. Namely, using the same notation as before and denoting by $U := X - Z$, there is a long exact sequence,

$$\cdots \to H^i_{Z,rig}(X) \to H^i_{rig}(X) \to H^i_{rig}(U) \to \cdots$$

The following result allows to compute the rigid cohomology with support on a closed subscheme using not all X but only an open subset:

Proposition 2.3.4. *Let X be a k-scheme of finite type, $Z \subset X$ a closed subscheme, and X' an open of X containing Z. Then, the canonical homomorphism*

$$H^*_{Z,rig}(X) \to H^*_{Z,rig}(X')$$

is an isomorphism.

Finally, we have a Gysin isomorphism. Using the same notation as in the previous proposition, this relates the cohomology of Z with the cohomology of X with support in Z.

Proposition 2.3.5. *If Z is of codimension r, then there is a Gysin isomorphism*

$$H^*_{rig}(Z) \xrightarrow{\sim} H^{*+2r}_{Z,rig}(X).$$

2.3.3 Rigid Cohomology with Compact Support

In this section we define rigid cohomology with compact support. Here our main references are [Be97i] and [Be97ii].

This notion shall allow to define rigid homology and to have Poincaré duality. We consider again a separated k-scheme X of finite type. We take a Nagata compactification $\overline{X}$ and we suppose that there exists an embedding in a formal $\mathcal{V}$-scheme. Denote $Z = \overline{X} - X$, and i the inclusion $]Z[_{\mathcal{P}} \hookrightarrow]\overline{X}[_{\mathcal{P}}$. For any abelian sheaf $\mathcal{F}$ on $]\overline{X}[_{\mathcal{P}}$, we define

$$\chi^0_{]X[}(\mathcal{F}) := \ker(\mathcal{F} \to i_* i^* \mathcal{F}).$$

This defines a left exact functor $\chi^0_{]X[}$ with right derived functors. Then, we define the *rigid cohomology with compact support of* X as

$$H^*_{c,rig}(X) := H^*(]\overline{X}[_{\mathcal{P}}, R\chi^0_{]X[}(\Omega^\bullet_{]\overline{X}[_{\mathcal{P}}})).$$

The following proposition states the basic properties of this cohomology.

Proposition 2.3.6. *1. There is a canonical homomorphism $H^i_{c,rig}(X) \to H^i_{rig}(X)$ that is an isomorphism if X is proper.*

2. $H^*_{c,rig}$ is a contravariant functor with respect to proper morphisms and covariant with respect to open immersions.

3. If X is an open of Y and $Z = Y - X$, then there is a long exact sequence

$$\cdots \to H^{i-1}_{c,rig}(Z) \to H^i_{c,rig}(X) \to H^i_{c,rig}(Y) \to H^i_{c,rig}(Z) \to H^{i+1}_{c,rig}(X) \to \cdots$$

Finally, this cohomology satisfies Poincaré duality. In general, if X is a k-scheme and $Z \subset X$ is a closed subscheme, there is a cup-product

$$H^i_{c,rig}(Z) \times H^j_{Z,rig}(X) \to H^{i+j}_{c,rig}(X),$$

which is functorial with respect to proper morphisms. If X is irreducible of dimension d, there is a trace map

$$\mathrm{tr} : H^{2d}_{c,rig}(X) \to K,$$

functorial with respect to open immersions. Now if X is smooth, by composition we get the Poincaré pairing

$$H^i_{c,rig}(Z) \times H^{2d-i}_{Z,rig}(X) \to H^{2d}_{c,rig}(X) \to K,$$

which is perfect. In particular, we get:

Proposition 2.3.7. *If X is a smooth k-scheme and $Z \subset X$ is a closed subscheme, then there is an isomorphism $H^i_{c,rig}(Z)^\vee \cong H^{2d-i}_{Z,rig}(X)$.*

2.3.4 Relative Log Crystalline Cohomology

In this section we recall the notions and main results of relative log crystalline cohomology following [Sh08]. We assume the reader familiar with PD-structures. The definition can be found for example in [BO78].

We begin with a field k of characteristic p, W a fixed Cohen ring of k and K the fraction field of W. We fix a p-adic fine formal log scheme $(\mathcal{B}, M_{\mathcal{B}})$ separated and topologically of finite type over Spf W. We denote by (B, M_B) its reduction modulo p.

Let $(X, M_X), (Y, M_Y)$ be fine log schemes over (B, M_B), and $(\mathcal{Y}, M_{\mathcal{Y}})$ a p-adic fine formal log scheme. Suppose that we are given a morphism $f : (X, M_X) \to (Y, M_Y)$ over (B, M_B) and the exact closed immersion $\iota : (Y, M_Y) \hookrightarrow (\mathcal{Y}, M_{\mathcal{Y}})$ given by the ideal $p\mathcal{O}_{\mathcal{Y}}$. In this situation we define the log crystalline site $(X/\mathcal{Y})^{log}_{crys}$:

Definition 2.3.1. An object of $(X/\mathcal{Y})^{log}_{crys}$ is $T := ((U, M_U), (T, M_T), i, \delta)$, where:

- (U, M_U) is a fine log scheme strict étale over (X, M_X),
- (T, M_T) is a fine log scheme over $(\mathcal{Y}, M_{\mathcal{Y}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}$ for some n,
- $i : (U, M_U) \hookrightarrow (T, M_T)$ is an exact closed immersion over $(\mathcal{Y}, M_{\mathcal{Y}})$,
- δ is a PD-structure on $\ker(\mathcal{O}_T \to i_*\mathcal{O}_U)$ which is compatible with the canonical PD-structure on $p\mathcal{O}_{\mathcal{Y}}$.

Morphisms are defined in a natural way and the coverings are the ones induced by étale coverings of T. We denote by $\mathcal{O}_{X/\mathcal{Y}}$ the sheaf on $(X/\mathcal{Y})^{log}_{crys}$ defined by $T \mapsto \Gamma(T, \mathcal{O}_T)$.

Given any site $\mathcal{S}$, denote by $\mathcal{S}^{\sim}$ the associated topos. Then, we have a natural functor $(X/\mathcal{Y})^{log,\sim}_{crys} \to \mathcal{Y}^{\sim}_{Zar}$ defined by

$$\mathcal{F} \mapsto (U \mapsto \Gamma((X \times_{\mathcal{Y}} U/U)^{log}_{crys}, \mathcal{F})),$$

and we denote its right derived functor (resp. qth right derived functor) by $Rf_{X/\mathcal{Y},crys,*}\mathcal{F}$ (resp. $R^q f_{X/\mathcal{Y},crys,*}\mathcal{F}$).

Remark 2.3.4. If we take $\mathcal{Y} = $ Spf W, with the log structure defined by $1 \mapsto 0$, we get the usual log crystalline cohomology $H^q_{log-crys}((X, M_X)/W^{\times})$

Now we define the notions of crystals and isocrystals:

Definition 2.3.2. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/\mathcal{Y}}$-modules on $(X/\mathcal{Y})^{log}_{crys}$ and for any object T of $(X/\mathcal{Y})^{log}_{crys}$, denote by $\mathcal{F}_T$ the induced sheaf on T_{Zar}. We say that $\mathcal{F}$ is a crystal if for any morphism $\varphi : T' \to T$ in $(X/\mathcal{Y})^{log}_{crys}$, the canonical morphism $\varphi^*\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism.

A crystal $\mathcal{F}$ is of finite presentation (resp. locally free of finite type) if for any object T of $(X/\mathcal{Y})^{log}_{crys}$, $\mathcal{F}_T$ is an $\mathcal{O}_T$-module of finite presentation (resp. a locally free $\mathcal{O}_T$-module of finite type). We denote by $C_{crys}((X/\mathcal{Y})^{log})$ the category of crystals of finite presentation on $(X/\mathcal{Y})^{log}_{crys}$. It is an abelian category.

For any abelian category $\mathcal{C}$, we can define a new category $\mathcal{C}_{\mathbb{Q}}$ as follows: the objects of $\mathcal{C}_{\mathbb{Q}}$ are the objects of $\mathcal{C}$ and for any two objects X, Y,

$$\mathrm{Hom}_{\mathcal{C}_{\mathbb{Q}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

When we consider X as an object of $\mathcal{C}_{\mathbb{Q}}$, we shall denote it by $X \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.3.3. We define the category of isocrystals on $(X/\mathcal{Y})^{log}_{crys}$ to be

$$I_{crys}((X/\mathcal{Y})^{log}) := C_{crys}((X/\mathcal{Y})^{log})_{\mathbb{Q}}.$$

For an isocrystal $\mathcal{E} = \mathcal{F} \otimes \mathbb{Q}$, we shall also use the notation $\mathcal{F} \otimes K$. We can define the relative log-crystalline cohomology of $(X, M_X)/(\mathcal{Y}, M_{\mathcal{Y}})$ with coefficient $\mathcal{E}$ as

$$Rf_{X/\mathcal{Y},crys,*}\mathcal{E} := Rf_{X/\mathcal{Y},crys,*}\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad R^q f_{X/\mathcal{Y},crys,*}\mathcal{E} := R^q f_{X/\mathcal{Y},crys,*}\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In our study we shall use mainly trivial coefficients, i.e., the trivial isocrystal that we denote by $\mathcal{O}_{X/\mathcal{Y},crys}$. Nevertheless, the results of this section shall be stated in a more general setting, in the lines of [Sh08].

Relative log crystalline cohomology can be computed using p-adically completed log PD-envelopes. Assume for the moment that (X, M_X) admits a closed immersion

$$(X, M_X) \hookrightarrow (\mathcal{P}, M_{\mathcal{P}})$$

into a p-adic fine log formal $\mathcal{B}$-scheme, which is formally smooth over $(\mathcal{Y}, M_{\mathcal{Y}})$. We denote by (D, M_D) the p-adically completed log PD-envelope of (X, M_X) in $(\mathcal{P}, M_{\mathcal{P}})$ and by $\mathrm{DR}(D/\mathcal{Y}, \mathcal{F})$ the log De Rham complex associated to the crystal $\mathcal{F}$ (the details of this construction can be found in section 1 of [Sh08]). Then, we have a quasi-isomorphism $Rf_{X/\mathcal{Y},crys,*}\mathcal{F} \cong Rf_*\mathrm{DR}(D/\mathcal{Y}, \mathcal{F})$. For an isocrystal $\mathcal{E} = \mathcal{F} \otimes \mathbb{Q}$, we define $\mathrm{DR}(D/\mathcal{Y}, \mathcal{E}) := \mathrm{DR}(D/\mathcal{Y}, \mathcal{F}) \otimes \mathbb{Q}$ and we get a quasi-isomorphism

$$Rf_{X/\mathcal{Y},crys,*}\mathcal{E} \cong Rf_*\mathrm{DR}(D/\mathcal{Y}, \mathcal{E}). \tag{2.2}$$

In the case where there is no closed immersion $(X, M_X) \hookrightarrow (\mathcal{P}, M_{\mathcal{P}})$, by [HK94] 2.18, there is always an embedding system

$$\begin{tikzcd}
(X^{(\bullet)}, M_{X^{(\bullet)}}) \arrow[r, "\iota"] \arrow[d, "g^{(\bullet)}"'] & (\mathcal{P}^{(\bullet)}, M_{\mathcal{P}^{(\bullet)}}) \\
(X, M_X) &
\end{tikzcd}$$

with $X^{(\bullet)} \to X$ an étale hypercovering and ι a closed immersion into a p-adic fine log formal $\mathcal{B}$-scheme, which is formally smooth over $(\mathcal{Y}, M_{\mathcal{Y}})$. Then, by the last part of page 7 in [Sh08], for any crystal $\mathcal{F}$ on $(X/\mathcal{Y})^{log}_{crys}$, we have

$$Rf_{X/\mathcal{Y},crys,*}\mathcal{F} = Rf_{X^{(\bullet)}/\mathcal{Y},crys,*}\mathcal{F}^{(\bullet)} = R(f \circ g^{(\bullet)})_*\mathrm{DR}(D^{(\bullet)}/\mathcal{Y}, \mathcal{F}^{(\bullet)}), \tag{2.3}$$

where $(D^{(\bullet)}, M_{D^{(\bullet)}})$ denotes the p-adically completed log PD envelope of

$$(X^{(\bullet)}, M_{X^{(\bullet)}})$$

in $(\mathcal{P}^{(\bullet)}, M_{\mathcal{P}^{(\bullet)}})$, and $\mathcal{F}^{(\bullet)}$ is the pull-back of $\mathcal{F}$ to $(X^{(\bullet)}, M_{X^{(\bullet)}})$.

Remark 2.3.5. The formulas 2.3 are still valid for isocrystals.

Now we state the main results regarding the structure of relative log crystalline cohomology. For the proofs, see theorems 1.15 and 1.16 of [Sh08].

Theorem 2.3.1. *Suppose f is proper and log smooth. Then, for any isocrystal $\mathcal{E}$, the relative log crystalline cohomology $R^q f_{X/\mathcal{Y},crys,*}\mathcal{E}$ is an isocoherent sheaf on $\mathcal{Y}$ for any $q \in \mathbb{N}$, i.e., a sheaf of $\mathbb{Q} \otimes \mathcal{O}_{\mathcal{Y}}$-modules on $\mathcal{Y}$ which is isomorphic to $\mathbb{Q} \otimes F$, for some coherent sheaf F of $\mathcal{O}_{\mathcal{Y}}$-modules. Moreover, $R^q f_{X/\mathcal{Y},crys,*}\mathcal{E}$ is zero for sufficiently large q.*

Theorem 2.3.2. *Assume that $\mathcal{E}$ is a locally free isocrystal and that either f is integral or Y is regular. Then, $Rf_{X/\mathcal{Y},crys,*}\mathcal{E}$ is a perfect complex of $\mathcal{O}_{\mathcal{Y}} \otimes \mathbb{Q}$-modules.*

To finish this section we state the following base-change theorem:

Theorem 2.3.3. *Assume we are given a diagram*

$$\begin{tikzcd}
(X', M_{X'}) \arrow[r] \arrow[d] & (Y', M_{Y'}) \arrow[r] \arrow[d] & (\mathcal{Y}', M_{\mathcal{Y}'}) \arrow[d, "\varphi"'] \\
(X, M_X) \arrow[r, "f"] & (Y, M_Y) \arrow[r, "\iota"] & (\mathcal{Y}, M_{\mathcal{Y}})
\end{tikzcd}$$

where f is proper, log smooth and integral, ι is the exact closed immersion defined by the ideal $p\mathcal{O}_{\mathcal{Y}}$ and the squares are cartesian. Then, for any locally free isocrystal $\mathcal{E}$ on $(X/\mathcal{Y})^{log}_{crys}$, we have the quasi-isomorphism

$$L\varphi^* Rf_{X/\mathcal{Y},crys,*}\mathcal{E} \xrightarrow{\sim} Rf_{X'/\mathcal{Y}',crys,*}\varphi^*\mathcal{E}.$$

The proof in general can be found in [Sh08], but as it is pointed out in remark 1.20 of that reference, for the case of isocrystals of the form $\mathcal{F} \otimes \mathbb{Q}$, where $\mathcal{F}$ is a locally free crystal of finite type, this result was essentially obtained in [BO78], 7.8.

2.4 Néron-Popescu Desingularization

In the next chapter we shall use a desingularization result proved by Popescu (theorem 1.8 in [Po]). An exposition of this result, its proof and some of its consequences is the paper of Swan [Sw95]. In this section we state only the definitions and results that we are going to use.

Definition 2.4.1. Let $f : R \to A$ be a ring homomorphism. Then, f is said to be formally smooth if for any R-algebra B with a nilpotent ideal $I \subset B$, and any R-algebra homomorphism $g : A \to B/I$, can be lifted to an R-algebra homomorphism $\widetilde{g} : A \to B$

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & A \arrow[d, "g"] \arrow[dl, dashed, "\widetilde{g}"] \\ B \arrow[r] & B/I \end{tikzcd}$$

Moreover, f is said to be formally étale if the morphism $\widetilde{g}$ is unique

Definition 2.4.2. Using the notation from the previous definition, we say that f is smooth if it is formally smooth and A is finitely presented over R (via f). Similarly, f is étale if it is formally étale and A is finitely presented over R.

Example 2.4.1. Any polynomial extension $R \to R[x_1, ..., x_n]$ is smooth. Moreover, let $f_1, ..., f_n \in R[x_1, .., x_n]$ and suppose that the jacobian $\Delta = \det |\partial f_i / \partial x_j|$ is invertible in $R[x_1, ..., x_n]/(f_1, ..., f_n)$. Then, $R \to R[x_1, ..., x_n]/(f_1, ..., f_n)$ is smooth.

Example 2.4.2. Let $f \in R[Y]$ be a monic polynomial in one variable. Then, $R \to (R[Y]/(f))_{f'}$ is étale.

We are going to need some extra definitions concerning geometrical regularity. We first define it for a local ring containing a field, and then for a ring homomorphism. This shall allow us to state the main theorem of this section.

Definition 2.4.3. Let R be a local ring and suppose that a field k of characteristic p is contained in R. We say that R is geometrically regular over k if for any finite field extension k'/k, such that $(k')^p \subset k$, the ring $k' \otimes_k R$ is regular.

Definition 2.4.4. A ring homomorphism $f : R \to A$ is geometrically regular if it is flat and for each prime ideal $\mathfrak{p}$ of R and each prime ideal $\mathfrak{q}$ of A lying over $\mathfrak{p}$, the ring $A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}} = k(\mathfrak{p}) \otimes_R A_{\mathfrak{q}}$ is geometrically regular over $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$.

Popescu's main theorem gives an equivalence condition for a morphism to be geometrically regular. We shall use this in order to consider a family of varieties defined over a ring of type $k[[t]]$ as part of a larger family.

Theorem 2.4.1. *Let $f : R \to A$ be a morphism of rings. Then, f is geometrically regular if and only if A is a filtered colimit of smooth R-algebras.*

The article [Sw95] is an exposition of the proof of this theorem.

Now we shall focus specifically in the case $A = k[[t]]$, for k a finite field, since this is the type of rings that we are going to use to prove our main theorem. Namely, it can be checked that the natural morphism $k[t] \to k[[t]]$ is geometrically regular:

Proposition 2.4.1. *The natural morphism $k[t] \to k[[t]]$ is geometrically regular.*

Proof. It is clearly flat, since it is a completion. Now there are only two prime ideals of $k[[t]]$. Namely, 0 and (t), and their respective counterpart in $k[t]$ are the only couples to consider in definition 2.4.4.

Case 1 (the ideal generated by t): in this case, we need to check that $k \otimes_{k[t]} k[[t]]_{(t)} \cong k$ is geometrically regular over k in the sense of definition 2.4.3, which is trivial.

Case 2 (the ideal 0): in this case, we need to check that $k(t) \otimes_{k[t]} k((t))$ is geometrically regular over $k(t)$, in the sense of definition 2.4.3. To do this, take a finite extension k' of $k(t)$ such that $(k')^p \subset k(t)$, and note that k' is necessarily $k(t^{1/p})$. Indeed, it is a finite extension of degree p (hence it does not have any subextension) and $(k')^p = k(t)$. Finally, since

$$k(t^{1/p}) \otimes_{k(t)} (k(t) \otimes_{k[t]} k((t))) \cong k((t^{1/p})),$$

we can conclude that $k(t^{1/p}) \otimes_{k(t)} (k(t) \otimes_{k[t]} k((t)))$ is a regular local ring, which completes the proof. □

In particular,

$$k[[t]] = \varinjlim_{\alpha} A_\alpha,$$

where the A_α's are smooth $k[t]$-algebras. One can say even more:

Proposition 2.4.2. *Let $\overline{k}$ be an algebraic closure of k and A a smooth $\overline{k}[t]$-algebra. Then, there exists a finite extension k' of k and a smooth $k'[t]$-algebra A' such that $A' \otimes_{k'} \overline{k} \cong A$.*

Proof. Let us take a presentation of A of the type $\overline{k}[t][x_1, ..., x_n]/(f_1, ..., f_c)$. Since $f_1, ..., f_c$ are a finite number of polynomials, one needs a finite number of elements of $\overline{k}$ to define them in the variables $t, x_1, ..., x_n$. Let k' be a finite extension of k containing all of those coefficients and define $A' := k'[t][x_1, ..., x_n]/(f_1, ..., f_c)$. We only need to assure that A' is smooth over $k'[t]$. This is a direct consequence of corollary 17.7.3, part ii), in EGAIV.

$\square$

This proposition implies that if we can deform the geometric special fiber of a proper scheme over $k[[t]]$, then we can do it for the base change to a finite extension k'/k. In particular, in this case we can still work with finite fields. This shall be useful to prove theorem 5.2.1.

Chapter 3

Generalities on $K3$ surfaces

The notion of $K3$ surface was introduced by A. Weil and this name was given in honor of the three geometers Kummer, Kähler and Kodaira. The purpose of this chapter is to get familiar with the properties of $K3$ surfaces, since this is the object treated in our main theorem.

In the first section, we study complex $K3$ surfaces and in the second we give a more general definition that works for p-adic fields. In the last section we shall see the results concerning Log $K3$ surfaces that will be useful to prove our main theorem.

3.1 $K3$ Surfaces over $\mathbb{C}$

Here we state only a few basic facts about the theory of complex $K3$ surfaces. A more complete treatment of this subject can be found, for example, in [BPV].

Definition 3.1.1. Let X be a compact smooth complex manifold of dimension 2. We shall say that X is a $K3$ surface if the following conditions are satisfied:

1. The canonical bundle ω_X is trivial.

2. The first Betti number $b_1(X) = 0$.

In [Siu], it is proven that with this definition, any $K3$ surface is a Kähler manifold. In particular, one can use Hodge Theory to study their cohomology. We shall do this in section 4.1, where we use the Clemens-Schmid exact sequence in order to get information on certain families of $K3$ surfaces.

The Betti numbers, and moreover the Hodge diamond of a complex $K3$ surface can be completely described:

Proposition 3.1.1. *Let X be a complex $K3$ surface. Then, $b_1(X) = b_3(X) = 0$, $b_0(X) = b_4(X) = 1$, $b_2(X) = 22$. Moreover, its Hodge diamond is the following:*

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Another important fact concerning the cohomology of $K3$ surfaces is the following

Proposition 3.1.2. *Let X be a complex $K3$ surface. Then, the cup-product $\cup$ induces a structure of non-degenerated lattice on $H^2(X, \mathbb{Z})$.*

This allows to reduce geometric problems into problems of lattices. A very important result on this is the following, known as the *weak Torelli's theorem*:

Theorem 3.1.1. *Let X, X' be complex $K3$ surfaces and suppose that there is an isometry $\phi : (H^2(X, \mathbb{Z}), \cup) \to (H^2(X', \mathbb{Z}), \cup)$ such that $\phi(H^{2,0}(X)) = H^{2,0}(X')$. Then, X is isomorphic to X'.*

The proof of this theorem can be found in [LP80].

3.2 $K3$ Surfaces over more General Fields

We are interested in the case of algebraic $K3$ surfaces. It is well-known from Hodge Theory that for complex algebraic surfaces, we have

$$q := \dim H^1(X, \mathcal{O}_X) = \frac{1}{2} b_1(X).$$

The number q is called *irregularity of X*. Thus, in this case the condition 2 of the definition is equivalent to have irregularity 0. This leads to the more general definition:

Definition 3.2.1. Let K be any field and X a smooth, proper algebraic variety over K of dimension 2. We shall say that X is a $K3$ surface if the following conditions are satisfied:

1. The canonical sheaf ω_X is trivial, i.e., isomorphic to $\mathcal{O}_X$.

2. The irregularity of X is 0.

This is the definition of $K3$ surface that we are interested in, since we want to work over p-adic fields. More specifically, we are interested in finite extensions K of $\mathbb{Q}_p$.

Remark 3.2.1. A direct consequence of the first condition of the definition is that the canonical divisor K_X is 0.

By Serre duality, we get that the Euler characteristic $\chi(\mathcal{O}_X)$ of a $K3$ surface X is 2. Then, we can compute the Euler number using Noether's formula:

$$e(X) = 12\chi(\mathcal{O}_X) - K_X^2 = 24.$$

We can also note that since $e(X)$ can be written as alternating sum of the Betti numbers (that in this case they are the rank of the ℓ-adic cohomology groups, for $\ell \neq p$), we get that the Hodge diamond with entries $h^i(X, \Omega_X^j)$ is the same as in the complex case. As a consequence of this, the Hodge spectral sequence

$$E_1^{p,q} = H^j(X, \Omega_X^j) \Longrightarrow H_{DR}^*(X)$$

degenerates at level 1.

Now we restrict ourselves to fields of characteristic $p > 0$. Suppose that k is a perfect field of characteristic $p > 0$, and we denote by $W = W(k)$ its ring of Witt vectors. Then, one may study the modules of crystalline cohomology of a $K3$ surface X over k and we have:

Proposition 3.2.1. *The W-modules of crystalline cohomology $H_{crys}^i(X/W)$ are free of rank $1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$, respectively.*

The proof can be found in [De81].

Now we may wonder if we can lift X to characteristic 0, i.e., if there exists some proper scheme over W such that its special fiber is X. In [De81], Deligne addressed this problem, using deformation functors.

Theorem 3.2.1 (Deligne)**.** *Let X be a $K3$ surface over k and $\mathcal{X}/S$ its universal deformation over W. Then, $S \cong Spf\, W[[t_1, ..., t_{20}]]$.*

In order to get the desired lifting, we restrict to certain kind of $K3$ surfaces.

Definition 3.2.2. Let X be a $K3$ surface over a perfect field k of characteristic $p > 0$. We say that X is ordinary if the height of its Brauer group is 1.

Proposition 3.2.2. *The following conditions are equivalent:*

1. *X is ordinary*
2. *The Frobenius $F : H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)$ is bijective.*
3. *The Hodge and Newton polygons of $H^2_{crys}(X/W)$ coincide.*
4. *$H^i(X, d\Omega^j_{X/k}) = 0$ for all i, j.*

For the proof of this, see Lemma 1.3 in [Ny83].

If k is moreover a finite field, Nygaard defined a formal lifting $\mathcal{X}_{can}$ over Spf W by pulling back the universal formal family $\mathcal{X}/S$ along the 0-section Spf $W \to S$. The formal scheme $\mathcal{X}_{can}$ is algebraizable, hence it defines a $K3$ surface lifting, denoted by X_{can}, over W. For the details, see proposition 1.8 of [Ny83].

Definition 3.2.3. X_{can} is called the canonical lifting of X.

A similar technique to the one used by Nygaard can be used for log $K3$ surfaces, which we treat in the following section.

3.3 Log $K3$ Surfaces

In this section we state the main results of [Na00]. This shall play a fundamental role in the proof of our main theorem. We begin by recalling the following:

Definition 3.3.1. Let k be a field and Y a variety over k. We say that Y/k is a normal crossing variety if Y is geometrically connected, the irreducible components of Y are geometrically irreducible and are of the same dimension d, and Y is a scheme over k that étale locally is isomorphic to

$$\text{Spec } k[x_0, ..., x_d]/(x_0 \cdots x_r).$$

Let m be the number of connected components of the singular locus Y_{sing} of Y. Let us denote them by D_i, for $i = 1, ..., m$, and assume they are geometrically connected. We endow Spec k with the log structure defined by $\mathbb{N}^m \to k$, as follows:

$$e_i \mapsto 0,$$

for all the canonical generators of $\mathbb{N}^m$. We denote this log-scheme by Spec $k^{\log}$ or as (Spec $k, \mathbb{N}^m$).

For each $i = 1, ..., m$, we can endow Spec $(k[x_0, ..., x_n]/(x_0 \cdots x_r))$ with a log structure given by as follows:

$$\mathbb{N}^{m+r} = \mathbb{N}^{i-1} \oplus \mathbb{N}^{r+1} \oplus \mathbb{N}^{m-i} \to k[x_0, ..., x_n]/(x_0 \cdots x_r)$$

$$e_i \mapsto \begin{cases} 0 & \text{if } e_i \in \mathbb{N}^{i-1} \\ x_{i-1} & \text{if } e_i \in \mathbb{N}^{r+1} \\ 0 & \text{if } e_i \in \mathbb{N}^{m-i} \end{cases}$$

Then,

1. If x is a smooth point of Y, étale locally on a neighbourhood of x, the log structure is the pull-back of the log structure of the log-point Spec $k^{\log}$

2. If $x \in D_i$, étale locally on a neighbourhood of x, the log structure is the pull-back of the log structure defined above.

Definition 3.3.2. We denote by $Y^{\log}$ the log scheme defined above and we call $Y^{\log}$/Spec $k^{\log}$ a normal crossing log (NCL) variety.

Remark 3.3.1. Note that we can also define a log structure in Spec k given by $\mathbb{N} \to k$, $1 \mapsto 0$. We denote this by (Spec $k, \mathbb{N}$). Then,

$$Y'^{\log} = Y^{\log} \times_{\text{Spec } k^{\log}} (\text{Spec } k, \mathbb{N}),$$

where $Y'^{\log}$ is the canonical log structure associated to a normal crossing variety (see [Ka96]).

Moreover, they give the same sheaf of relative log differentials

$$\omega^1_{Y/k} = \omega^1_{Y'/k}.$$

Definition 3.3.3. Let $Y^{\log}/\mathrm{Spec}\ k^{\log}$ be a NCL variety. We say that it is a simple normal crossing log (SNCL) variety if the underlying scheme Y is a simple normal crossing variety, i.e., its irreducible components are smooth and geometrically irreducible.

As one can notice, these definitions are stated for varieties of any dimension, but we want to focus our attention to the case of surfaces, and more concretely to the case of log $K3$ surfaces:

Definition 3.3.4. Let $X^{\log}/\mathrm{Spec}\ k^{\log}$ be a NCL variety of pure dimension 2. We say that $X^{\log}/\mathrm{Spec}\ k^{\log}$ is a normal crossing log $K3$ surface if the underlying scheme is a proper scheme over $\mathrm{Spec}\ k$, such that $H^1(X, \mathcal{O}_X) = 0$ and $\omega^2_{X/k} \cong \mathcal{O}_X$.

Another definition that plays a central role en route to the proof of our main theorem is that of combinatorial $K3$ surface.

Definition 3.3.5. Let X be a proper surface over a field k. Let $\overline{k}$ be an algebraic closure of k. Consider the following conditions:

I) X is a smooth $K3$ surface over k.

II) $X \otimes_k \overline{k} = X_1 \cup X_2 \cup \cdots X_N$ is a chain of smooth surfaces with X_1, X_N rational and the others elliptic ruled, such that the double curves on each of them being rulings.

III) $X \otimes_k \overline{k} = X_1 \cup X_2 \cup \cdots X_N$, with every X_i a rational surface, the double curves on X_i are rational and form a cycle on X_i. The dual graph of $X \otimes_k \overline{k}$ is a triangulation of the sphere S^2.

The surface X is called a combinatorial Type I (Type II or Type III, respectively) $K3$ surface if X satisfies I (II or III, respectively), X has a log structure whose charts are given by its local normal crossing components as in definition 3.3.2, and $\omega^2_{X/k} \cong \mathcal{O}_X$.

In general we shall refer to a combinatorial $K3$ surface when it is not necessary to specify which type is. Now we give two important results about combinatorial $K3$ surfaces. They are theorem 3.3 and proposition 3.4 in [Na00], respectively.

Proposition 3.3.1. *Let X be a combinatorial Type II or Type III $K3$ surface over k. Then,* $\Gamma(X, \omega^1_{X/k}) = 0$.

Proposition 3.3.2. *Let $X^{\log}/Spec\ k^{\log}$ be an SNCL $K3$ surface. Then, $X \otimes_k \overline{k}$ is a combinatorial $K3$ surface.*

Since we shall deal with $K3$ surfaces X_K over a p-adic field K, having a semistable model X, we shall use the following:

Proposition 3.3.3. *If X_K has a minimal semistable model X, then its special fiber is automatically an SNCL $K3$ surface.*

This result can be found in page 20 of [Mau12]. The following is a consequence of the works by Kawamata (beginning of section 1 in [Kw93], and section 3 in [Kw98]), which together with propositions 3.3.3 and 3.3.2 tells us that if $p > 3$, we can always assume that the special fiber is a combinatorial $K3$ surface.

Proposition 3.3.4 (Kawamata)**.** *If X_K has a semistable model and $p > 3$, then X_K has a minimal semistable model.*

Now we consider the log-analogue of the notion of ordinary $K3$ surface from the preceding section. Suppose that k is perfect of characteristic $p > 0$, and let $f^{\log} : Y^{\log} \to \mathrm{Spec}\ k^{\log}$ be a log smooth, integral morphism of Cartier type of fine log schemes such that f is proper.

Definition 3.3.6. $f^{\log}$ (or $Y^{\log}$) is said to be log ordinary if $H^j(Y, d\omega^i_{Y/k}) = 0$ for all i, j.

One can find log-versions of the equivalences from proposition 3.2.2. In particular, one can define a log enlarged formal Brauer group of $X^{\log}$ and the condition of being log ordinary is equivalent to this having height 1. This is important when one wants to have a log-lifting in characteristic 0, since one might proceed as in the preceding section, following what Nygaard did for ordinary $K3$ surfaces and do the log analogue of his proof for log ordinary $K3$ surfaces. This is what Nakkajima does in section 5 of [Na00], and this theorem is obtained:

Theorem 3.3.1. *Assume that $X^{\log}$ is log ordinary. If X is projective, there exists a log deformation $\mathfrak{X}^{\log}_{can}$ over Spec $W[[u_1, ..., u_m]]^{\log}$.*

We can get a log deformation over Spec $W^{\log}$ by specializing each of the variables u_i to p, and a log deformation over Spec $k[[t]]^{\log}$ by specializing each of the variables to t and reducing modulo p.

Now we have the following result that tells us that the log ordinary assumption is not too restrictive:

Proposition 3.3.5. *An SNCL $K3$ surface $X^{\log}$ of Type II is log ordinary if and only if the double elliptic curve is ordinary. Any SNCL $K3$ of Type III is log ordinary.*

However, one would like to get rid of the log-ordinary assumption, in order to get these deformations for a more general situation. This is possible, but we need to assume that the base field k is algebraically closed. This is what Nakkajima does in section 6 of [Na00]. Then, we have the following:

Theorem 3.3.2 (Nakkajima)**.** *Let k be an algebraically closed field of characteristic $p > 0$. Let $X^{\log}$ be a projective SNCL $K3$ surface over Spec $k^{\log}$. Then, there exists a log smooth family $\mathfrak{X}^{\log}$ over Spec $W[[u_1, ..., u_m]]^{\log}$ that is a charted deformation of $X^{\log}$.*

And by proceeding as after theorem 3.3.1, we get the following:

Corollary 3.3.1. *Let X be a projective SNCL $K3$ surface over k (algebraically closed). The following hold:*

1) There exists a projective semistable family $\widetilde{X}$ over Spec W whose special fiber is X.

2) There exists a projective semistable family $\widetilde{X}$ over Spec $k[[t]]$ whose special fiber is X.

Chapter 4

Clemens-Schmid exact sequence over a local basis

In a classical situation (that we describe in the first section of this chapter), given a semistable degeneration over the complex unit disk $\pi : X \to \Delta$, the Clemens-Schmid exact sequence relates the topology and Hodge Theory of the central fiber to that of a generic (smooth) fiber, by means of the monodromy of the family π^* restricted to the punctured disk.

One may wonder if it is possible to do a p-adic version of this. A first arithmetic analogue to a semistable degeneration is a family of varieties in characteristic $p > 0$ defined over a smooth curve C, and such that a "special" fiber is a NCD. This geometric situation was studied by Chiarellotto and Tsuzuki and they obtained a Clemens-Schmid type exact sequence. In the second section of this chapter we make a brief description of their work and their main result.

The next natural arithmetic analogue is to consider a family of varieties defined over Spec $\mathcal{V}$, where $\mathcal{V}$ is a complete discrete valuation ring. This is usually understood as an arithmetic analogue of the complex unit disk. This new problem can have different variants: mixed characteristic, equicharacteristic $p > 0$, equicharacteristic 0. The main purpose of this chapter is to obtain a Clemens-Schmid type exact sequence for one of this cases: equicharacteristic $p > 0$. Indeed, we may assume that $\mathcal{V} = k[[t]]$, and we shall use Néron-Popescu desingularization to reduce this problem to the one that Chiarellotto and Tsuzuki studied. In sections 4.3-4.6 we do this.

4.1 Classical Clemens-Schmid exact sequence

Consider the following classical situation: let Δ be the open disk around 0 in the complex plane and X a smooth complex variety of dimension $n+1$, with Kähler total space. Suppose that $\pi : X \to \Delta$ is a semi-stable degeneration: a holomorphic, proper and flat map such that π is smooth outside the central fiber $X_0 = \pi^{-1}(0)$, which is a normal crossing divisor, i.e., it is a sum of irreducible components meeting transversally, and such that each of them is smooth.

In this case, the restriction of π to the punctured disc $\pi^* : X^* \to \Delta^*$ is a C^∞ fibration, so $\pi_1(\Delta^*)$ acts on the cohomology $H^m(X_t)$ of the fiber $X_t = \pi^{-1}(t)$, for any $t \neq 0$. The *Picard-Lefschetz transformation*, denoted by $T : H^m(X_t) \to H^m(X_t)$ is the map induced by the canonical generator of $\pi_1(\Delta^*)$. Then, one can prove (see for example the appendix of [La73]) that T is unipotent, i.e., $(T-I)^{m+1} = 0$, where I is the identity operator. This allows to define a monodromy operator as

$$N := \log T = (T-I) - \frac{1}{2}(T-I)^2 + \frac{1}{3}(T-I)^3 - \cdots ,$$

which is in fact a finite sum. It is also clear that N is nilpotent, hence we can endow $H^m(X_t)$ with an increasing filtration

$$0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m} = H^m(X_t)$$

which is the unique filtration such that:

1) $N(W_k) \subset W_{k-2}$.

2) N^k induces an isomorphism on the graded parts:

$$Gr_{m+k}(H^m(X_t)) \xrightarrow{\sim} Gr_{m-k}(H^m(X_t)).$$

One can make an explicit description of this filtration. See for example section 2 of [Mo84].

We define a filtration on $H^m(X_0)$ via a spectral sequence. Denote by $Y_1, ..., Y_r$ the irreducible components of X_0, which we assumed to be smooth and proper. We define the *codimension p stratum of X_0* as

$$X^{[p]} := \bigsqcup_{i_0<\cdots<i_p} Y_{i_0} \cap \cdots \cap Y_{i_p}.$$

We define $E_0^{p,q} = A^q(X^{[p]})$, the C^∞ q-forms on $X^{[p]}$. Then, we have $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ the exterior derivative. We also have morphisms $\delta_0^{p,q} : E_0^{p,q} \to E_0^{p+1,q}$ induced by the combinatorial formula

$$(\delta_0^{p,q}\omega)|_{Y_{j_0}\cap\cdots\cap Y_{j_{p+1}}} = \sum_{k=0}^{p+1}(-1)^k \omega|_{Y_{j_0}\cap\cdots\cap\widehat{Y}_{j_k}\cap\cdots\cap Y_{j_{p+1}}}$$

where $\widehat{Y}_{j_k}$ means that we ignore this term. This defines a double complex $(E_0^{\bullet,\bullet}, d, \delta)$ and we have the following:

Theorem 4.1.1. *The spectral sequence with*

$$E_1^{p,q} = H^q(X^{[p]})$$

degenerates at level 2 and it converges to $H^*(X_0)$.

By letting

$$W_k = \bigoplus_{q\leq k} E_0^{*,q},$$

we get a filtration on the simple complex associated to the double complex $(E_0^{\bullet,\bullet}, d, \delta)$, and consequently a filtration on $H^m(X_0)$.

One can construct a retraction $r : X \to X_0$ which induces isomorphisms

$$r^* : H^m(X_0) \xrightarrow{\sim} H^m(X) \tag{4.1}$$

$$r_* : H_m(X) \xrightarrow{\sim} H_m(X_0). \tag{4.2}$$

The details of this construction are in [Cl77].

The isomorphism (4.2) allows to define a filtration on $H_m(X_0) \cong H_m(X) =: H_m$. Indeed, we use Poincaré duality and define

$$W_{-k}(H_m) = \mathrm{Ann}(W_{k-1}(H^m)) = \{h \in H_m | (W_{k-1}(H^m), h) = 0\}.$$

Now that we have filtrations on H^m, H_m and $H^m_{\mathrm{lim}} := H^m(X_t)$. We shall define maps relating them and respecting the filtrations in the following sense:

Definition 4.1.1. Let H, H' vector spaces with filtrations that we denote by $W_\bullet$ for both. A morphism of filtered vector spaces of type r is a linear map $\phi : H \to H'$ such that for all k,

$$\phi(W_k(H)) = W_{k+2r}(H') \cap \mathrm{Im}(\phi).$$

Recall that $\dim X = n + 1$. Then, we define a morphism $\alpha : H_{2n+2-m} \to H^m$ as the composite

$$H_{2n+2-m}(X_0) \xrightarrow{p} H^m(X, X - X_0) \longrightarrow H^m(X),$$

where p is the Poincaré duality map, and the second is the natural morphism.

We define $\beta : H^m_{\mathrm{lim}} \to H_{2n-m}$ as the composite

$$H^m(X_t) \xrightarrow{p_t} H_{2n-m}(X_t) \xrightarrow{i_*} H_{2n-m}(X),$$

where i_* is induced by the natural inclusion $X_t \hookrightarrow X$ and p_t is the Poincaré duality morphism. Then, we have the following:

Theorem 4.1.2 (Clemens-Schmid)**.** *The maps α, i^*, N, β are morphisms of filtered vector spaces of type $n + 1, 0, -1, -n$, respectively, and the sequence*

$$\cdots \to H_{2n+2-m} \xrightarrow{\alpha} H^m \xrightarrow{i^*} H^m_{\mathrm{lim}} \xrightarrow{N} H^m_{\mathrm{lim}} \xrightarrow{\beta} H_{2n-m} \xrightarrow{\alpha} H^{m+2} \to \cdots \tag{4.3}$$

is exact.

Remark 4.1.1. One can state the Clemens-Schmid exact sequence as an exact sequence of Mixed Hodge Structures. This explains the notation of H^m_{lim}, since that term is considered with the limit Mixed Hodge structure, defined by Steenbrink in [St76]. In fact, in order to prove that the sequence 4.3 is exact, one really needs to use the Mixed Hodge Structures involved, but one can get some applications by considering it only as an exact sequence of filtered vector spaces.

Since we are interested in studying surfaces, now we want to restrict ourselves to the case $n = 2$. By using the exact sequence (4.3) for H^2, restricted to the elements of the filtrations on each term, and the properties of the graded parts, one can prove the following *monodromy criteria*:

Theorem 4.1.3. *Let Γ be the dual graph of X_0 and denote*

$$\Phi = \dim\ker(H^1(X^{[0]}) \to H^1(X^{[1]})),$$

$$q = \frac{1}{2}h^1(X^{[0]}), \quad g = \frac{1}{2}h^1(X^{[1]}).$$

Then,

1. *$N = 0$ on H^1_{lim} if and only if $h^1(|\Gamma|) = 0$ if and only if $b_1(X_t) = \Phi$.*
2. *$N^2 = 0$ on H^2_{lim} if and only if $h^2(|\Gamma|) = 0$.*
3. *$N = 0$ on H^2_{lim} if and only if $h^2(|\Gamma|) = 0$ and $\Phi + 2g = 2q$.*

The proof of this theorem is in section 4(c) of [Mo84]. Now we apply these monodromy criteria to the case of semistable degenerations of $K3$ surfaces. In order to use it, we first need the following classification:

Theorem 4.1.4 (Kulikov)**.** *A semistable degeneration of $K3$ surfaces is birational to one for which the central fiber X_0 is one of three types:*

- *Type I. X_0 is a smooth $K3$ surface.*
- *Type II. $X_0 = Y_0 \cup \cdots \cup Y_{k+1}$, where Y_α intersects only $Y_{\alpha\pm1}$, and each $Y_\alpha \cap Y_{\alpha+1}$ is an elliptic curve. The surfaces Y_0, Y_{k+1} are rational, and for $1 \leq \alpha \leq k$, Y_α is ruled with $Y_\alpha \cap Y_{\alpha+1}$ and $Y_\alpha \cap Y_{\alpha-1}$ sections of the ruling.*
- *Type III. All components of X_0 are rational surfaces, $Y_i \cap (\cup_{j\neq i} Y_j)$ is a cycle of rational curves, and $|\Gamma| = S^2$.*

Now a simple application of the criteria from theorem 4.1.3 to these three cases, we get a classification of the special fiber in terms of the monodromy operator N:

Theorem 4.1.5. *1. X_0 is of type I if and only if $N = 0$ on $H^2_{\lim}$.*

2. X_0 is of type II if and only if $N \neq 0$, but $N^2 = 0$ on $H^2_{\lim}$.

3. X_0 is of type III if and only if $N^2 \neq 0$.

4.2 Chiarellotto-Tsuzuki's work

In this section we deal with a first analogue in characteristic $p > 0$, to the geometric situation in which the Clemens-Schmid exact sequence from the preceding section can be obtained. We make a brief description of Chiarellotto and Tsuzuki's work on this. The details can be found in [CT12].

Let k be a finite field of characteristic $p > 0$, X a smooth variety of dimension $n+1$ and C a smooth curve over k. Consider a proper and flat morphism $f : X \to C$, over k, and suppose moreover that for a k-rational point $s \in C$, the fiber of f at s, denoted by X_s is a normal crossing divisor (NCD) inside X, and that f is smooth outside X_s.

The NCD X_s allows to define a log structure M on X, and the NCD given by the point s allows to define a log structure N on C. We denote by $s^\times$ the log point given by the point s with the log structure induced from (C, N) via the closed immersion $\{s\} \hookrightarrow C$. By taking the fiber product, we get the following cartesian diagram:

$$\begin{tikzcd}
(X_s, M_s) \arrow[r] \arrow[d] & (X, M) \arrow[d, "f"] \\
s^\times \arrow[r] & (C, N)
\end{tikzcd} \tag{4.4}$$

where f is log-smooth.

Let $\mathcal{V}$ be a complete and absolutely unramified discrete valuation ring of mixed characteristic with residue field k, and fraction field K. We denote by $\mathcal{V}^\times$ the spectrum of $\mathcal{V}$ endowed with the log structure associated to $1 \mapsto 0$.

In this setting, one can construct a Clemens-Schmid type exact sequence, which is analogue to 4.3. To do this, one considers a smooth lifting $C_{\mathcal{V}}$ over $\mathcal{V}$ of C [1] and its p-

[1] One can always do this by 7.4 III in SGA 1.

adic completion $\mathcal{C}$. Then, by fixing a lift $\hat{s}$ of s in $\mathcal{C}$ and a local coordinate t of $\mathcal{C}$ over $\mathcal{V}$, we can define a log structure $\mathcal{N}$ on $\mathcal{C}$ by $1 \mapsto t$. Then, we have the following diagram:

$$\begin{tikzcd}
(X_s, M_s) \arrow[r] \arrow[d] & (X, M) \arrow[d, "f"] & \\
s^{\times} \arrow[r] & (C, N) \arrow[r] & (\mathcal{C}, \mathcal{N})
\end{tikzcd}$$

Recall that in this setting Shiho defined the sheaves of relative log crystalline cohomology, as in section 2.3.4. Moreover, one can find relations between these and the sheaves of relative log convergent and log analytic cohomology. In [CT12] they consider only the situation that they are interested in (where the basis are curves), but these remain valid in more general situations as we state in section 4.5.

One can define a Frobenius structure on the sheaves of relative cohomology and a log integrable connection. This allows to define a monodromy operator N_m on the K-vector space $H^m_{log-crys}((X_s, M_s)/\mathcal{V}^{\times}) \otimes K$, that coincides with the one defined by Hyodo-Kato in [HK94].

The p-adic analogue of 4.3 is constructed by putting together two long exact sequences that are compatible with Frobenius structures, since these give the weight filtration. Namely,

$$\cdots \to H^m_{log-conv}((X_s, M_s)/\mathcal{V}) \to H^m_{log-crys}((X_s, M_s)/\mathcal{V}^{\times}) \otimes K \xrightarrow{N_m}$$

$$H^m_{log-crys}((X_s, M_s)/\mathcal{V}^{\times}) \otimes K(-1) \to H^{m+1}_{log-conv}((X_s, M_s)/\mathcal{V}) \to \cdots, \tag{4.5}$$

and

$$\cdots \to H^m_{X_s,rig}(X) \to H^m_{rig}(X_s) \to H^m_{log-conv}((X_s, M_s)/\mathcal{V}) \to H^{m+1}_{X_s,rig}(X) \to \cdots \tag{4.6}$$

and the following sequence is obtained:

$$\cdots \to H^m_{rig}(X_s) \xrightarrow{\gamma} H^m_{log-crys}((X_s, M_s)/\mathcal{V}^{\times}) \otimes K \xrightarrow{N_m}$$

$$H^m_{log-crys}((X_s, M_s)/\mathcal{V}^{\times}) \otimes K(-1) \xrightarrow{\delta} H^{m+2}_{X_s,rig}(X) \xrightarrow{\alpha} H^{m+2}_{rig}(X_s) \to \cdots, \tag{4.7}$$

Once that the sequence 4.7 is constructed, one needs to prove that it is exact. As in the classical case, the most important point is the equivalence of the weight filtration and the monodromy filtration on log-crystalline cohomology. This is done by comparing the log-crystalline cohomology to the log-analytic and the rigid one. Then, it is a direct consequence of theorem 10.8 in Crew's article [Cr98].

The equivalence of the weight and monodromy filtration allows to use weights arguments to prove the following:

Theorem 4.2.1 (Chiarellotto-Tsuzuki)**.** *The sequence 4.7 is exact.*

With this long exact sequence in hand, we can prove another p-adic analogue of 4.3. Namely, for a family of varieties over a local basis, which is the main purpose of this chapter, and particularly of the next sections. We shall see that it can be obtained as a consequence of theorem 4.2.1.

4.3 Notation and setting over a local basis

In this and the following sections of this chapter, we consider another analogue situation to the classical one. Namely, let k be a finite field of characteristic $p > 0$, and consider a proper and flat morphism

$$F : X \to \mathrm{Spec}\ k[[t]],$$

where X is smooth over k, such that étale locally is étale over

$$\mathrm{Spec}\ (k[[t]][x_1, ..., x_n]/(x_1 \cdots x_r - t)).$$

For this situation, we shall obtain an arithmetic version of the Clemens-Schmid exact sequence, similar to the one in [CT12]. In fact, we shall use that sequence, and for this purpose we shall see the special fiber of F as a fiber of a family over a smooth curve.

We fix a finite field of characteristic $p > 0$, denoted by k. We denote by $W = W(k)$ its ring of Witt vectors and K the fraction field of W. We shall denote by the same letter W the formal scheme Spf W with the trivial log structure, and we denote by $W^\times$ the same formal scheme with the log structure given by $1 \mapsto 0$.

Recall that a divisor $Z \subset Y$ of a noetherian scheme is said to be a strict normal crossing divisor (SNCD) if Z is a reduced scheme and, if Z_i, $i \in J$ are the irreducible components of Z, then, for any $I \subset J$, the intersection $Z_I = \cap_{i \in I} Z_i$ is a regular scheme of codimension equal to the number of elements of I. We shall say that Y is a normal crossing divisor (NCD) if, étale locally on Y, it is a SNCD.

We consider a proper and flat morphism

$$F : X \to \text{Spec } k[[t]]$$

over k, where X is a smooth scheme such that étale locally it is étale over

$$\text{Spec}\left(k[[t]][x_1, ..., x_n]/(x_1 \cdots x_r - t)\right).$$

We denote by s the closed point of Spec $k[[t]]$ and X_0 its fiber, which is a NCD inside X. We denote by (X, M) the scheme X endowed with the log structure defined by X_0, Spec $k[[t]]^\times$ the scheme Spec $k[[t]]$ endowed with the log structure defined by the point s (i.e., by the NCD given by the ideal generated by t), and $s^\times$ the log point given by the point s and the log structure induced from Spec $k[[t]]^\times$. Then, we have the following cartesian diagram of log schemes

$$\begin{tikzcd} (X_0, M_0) \arrow[r] \arrow[d] & (X, M) \arrow[d, "F"] \\ s^\times \arrow[r] & \text{Spec } k[[t]]^\times \end{tikzcd} \tag{4.8}$$

where (X_0, M_0) is obtained by taking the fiber product in the category of log schemes.

4.4 A construction using Néron-Popescu desingularization

In order to get the desired result, we need to study the cohomology of the special fiber X_0 of X over $k[[t]]$, and for this, we use the results from section 2.4. In particular, recall that the ring $k[[t]]$ is a filtered colimit of smooth $k[t]$-algebras.

Since X is proper over $k[[t]]$, there exist a smooth $k[t]$-algebra A, a scheme X_A, proper over Spec A, Zariski locally étale over Spec $A[x_1, ..., x_n]/(x_1 \cdots x_r - t)$, and such that the following diagram is cartesian:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "F"'] & X_A \arrow[d, "f"] \\ \text{Spec } k[[t]] \arrow[r, "v"'] & \text{Spec } A \end{tikzcd}$$

Note that the composition $v \circ F$ is flat, hence f is flat in an open of X_A containing the image of X under u. Thus, we may assume that $f : X_A \to \text{Spec } A$ is flat.

Since the divisor of $Y = \text{Spec}A$, defined by $Y_0 = (t = 0)$ is a NCD, and the fiber product $X_{A,t=0} = Y_0 \times_Y X_A$ is a NCD divisor in X_A, then we can naturally define fine log structures M_A and N on X_A and Y, respectively. Then, $f : (X_A, M_A) \to (Y, N)$ is a morphism of log schemes. Moreover, we have the following:

Lemma 4.4.1. *The morphism $f : (X_A, M_A) \to (Y, N)$ is log-smooth.*

Proof. We use the criterion given in proposition 2.1.1. First note that f has (étale locally on X_A) a chart $(P_{X_A} \to M_A, Q_Y \to N, Q \to P)$ given by $Q = \mathbb{N}$, $P = \mathbb{N}^r$, and the diagonal map $Q \to P$.

We can easily see also that the kernel and the torsion part of the cokernel of $Q^{gp} \to P^{gp}$ (which is just the diagonal map $\mathbb{Z} \to \mathbb{Z}^r$) are both trivial.

It remains to prove that the induced morphism $X_A \to Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$ is smooth. Recall that X_A is locally étale over $V = \text{Spec } (A[x_1, ..., x_n]/(x_1 \cdots x_r - t))$, and note that

$$W = \text{Spec } A \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P] \cong \text{Spec } A \times_{\text{Spec } \mathbb{Z}[u]} \text{Spec } \mathbb{Z}[u_1, ..., u_r]$$

$$\cong \text{Spec } (A[u_1, ..., u_r]/(u_1 \cdots u_r - t)).$$

The last isomorphism can be verified by checking directly that the ring

$$A[u_1, ..., u_r]/(u_1 \cdots u_r - t)$$

satisfies the universal property of the tensor product $A \otimes_{\mathbb{Z}[u]} \mathbb{Z}[u_1, ..., u_r]$.

Now note that there are natural closed immersions $j_V : V \hookrightarrow \mathbb{A}^n_A$, $j_W : W \hookrightarrow \mathbb{A}^r_A$. Moreover, the following diagram is cartesian:

$$\begin{tikzcd} V \arrow[r, "j_V"] \arrow[d, "h"'] & \mathbb{A}^n_A \arrow[d, "p"] \\ W \arrow[r, "j_W"'] & \mathbb{A}^r_A \end{tikzcd}$$

where h is defined by sending each u_i to x_i for $i = 1, ..., r$, and p is the natural projection from the first r components. Since p is smooth, we get that h is smooth. Since $X_A \to W$ is the composition of an étale and a smooth morphism, we conclude that it is smooth (in the classical sense). $\square$

Then, we have the following diagram of log schemes:

$$\begin{tikzcd} (X_0, M_0) \arrow[r] \arrow[d, "f_s"'] & (X, M) \arrow[r] \arrow[d, "F"'] & (X_A, M_A) \arrow[d, "f"] \\ s^{\times} \arrow[r] & \mathrm{Spec}\ k[[t]]^{\times} \arrow[r] & (Y, N) \end{tikzcd}$$

where the horizontal arrows are exact closed immersions. In particular, note that s is a closed point inside Y, hence (X_0, M_0) is a fiber of the log smooth family $(X_A, M_A) \to (Y, N)$. This means that we can study the cohomology of X_0 using relative cohomology sheaves for this family. These are studied in the next section.

4.5 Relative Cohomology

Recall that A is a smooth $k[t]$-algebra. By theorem 7, in section 4 in [El73], there exists a $W[t]$-algebra A_0 such that $A_0/pA_0 = A$, which is smooth over W.[2] Let $\hat{A}$ be the p-adic completion of A_0, and $\mathcal{Y} = \mathrm{Spf}\ \hat{A}$. We can define a log structure $\mathcal{N}$ on $\mathcal{Y}$ by $1 \mapsto t$, and

[2]Note that A_0 might be not smooth over $W[t]$.

then we have the following diagram:

$$
\begin{tikzcd}
(X_0, M_0) \arrow[r] \arrow[d, "f_s"'] & (X_A, M_A) \arrow[d, "f"] & \\
s^\times \arrow[r] & (Y, N) \arrow[r] & (\mathcal{Y}, \mathcal{N})
\end{tikzcd}
\tag{4.9}
$$

where the lower row consists of two exact closed immersions. Now we are in the situation studied in [Sh08], and we can use all the results there. We shall state the results on relative log crystalline, log convergent and log analytic cohomology that are useful to apply the main result in [CT12], i.e., a Clemens-Schmid exact sequence in characteristic p.

Relative Log Crystalline Cohomology

In the situation of diagram (4.9), Shiho defined in [Sh08], for any sheaf $\mathcal{F}$ on the log crystalline site $(X/\mathcal{Y})^{log}_{crys}$ the sheaves of relative log crystalline cohomology of $(X_A, M_A)/(\mathcal{Y}, \mathcal{N})$ with coefficient $\mathcal{F}$, denoted by $R^m f_{X_A/\mathcal{Y},crys*}\mathcal{F}$, and for an isocrystal $\mathcal{E} = \mathbb{Q} \otimes \mathcal{F}$, denoted by $R^m f_{X_A/\mathcal{Y},crys*}\mathcal{E}$. Here we will work only with the trivial log isocrystal $\mathcal{E} = \mathcal{O}_{X_A/\mathcal{Y},crys}$.

In order to study the sheaves $R^m f_{X_A/\mathcal{Y},crys*}\mathcal{O}_{X/\mathcal{Y},crys}$, we fix a Hyodo-Kato embedding system $(\mathcal{P}_\bullet, \mathcal{M}_\bullet)$ of an étale hypercovering $(X_\bullet, M_\bullet)$ of (X_A, M_A). It always exists, as stated in [HK94] 2.18 (the definition of simplicial schemes and étale hypercoverings can be found in [CT03]). Then, we have the following diagram:

$$
\begin{tikzcd}
(X_{0,\bullet}, M_{0,\bullet}) \arrow[r] \arrow[d, "\theta_s"'] & (X_\bullet, M_\bullet) \arrow[r, "i_\bullet"] \arrow[d, "\theta"'] & (\mathcal{P}_\bullet, \mathcal{M}_\bullet) \arrow[dd, "g"] \\
(X_0, M_0) \arrow[r] \arrow[d, "f_s"'] & (X_A, M_A) \arrow[d, "f"] & \\
s^\times \arrow[r] & (Y, N) \arrow[r] & (\mathcal{Y}, \mathcal{N})
\end{tikzcd}
\tag{4.10}
$$

where $(X_{0,\bullet}, M_{0,\bullet})$ is the fiber product in the upper left square.

We want to see that the sheaves $R^m f_{X_A/\mathcal{Y},crys*}(\mathcal{O}_{X/\mathcal{Y},crys})$ satisfy some finiteness properties. For each $n \in \mathbb{N}$, denote by $\mathcal{Y}_n$ the reduction of $\mathcal{Y}$ modulo p^n, and $C_{X_\bullet/\mathcal{Y}_n}$ the logarithmic De Rham complex of the log PD-envelope of the closed immersion $i_\bullet$ over $(\mathcal{Y}_n, \mathcal{N}_n)$. Then, we have the following:

Lemma 4.5.1. *(a) For each n, there is a canonical quasi-isomorphism*

$$R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n} \otimes^L_{\mathcal{O}_{\mathcal{Y}_n}} \mathcal{O}_{\mathcal{Y}_{n-1}} \xrightarrow{\sim} R(f\theta)_* C_{X_\bullet/\mathcal{Y}_{n-1}}.$$

(b) For each n, $R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n}$ is bounded and has finitely generated cohomologies.

Proof. In section 1 of [Sh08], it is proved that

$$R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n} \cong R f_{X_\bullet/\mathcal{Y}_n,crys,*}(\mathcal{O}_{X_\bullet/\mathcal{Y}_n,crys}),$$

and so part (a) follows from the claim in the proof of theorem 1.15, in [Sh08].

For part (b), we proceed inductively. Note that for $n = 1$, $\mathcal{Y}_1 = Y$, and so the result follows by properness of f. The inductive step is direct by using the second part of the same claim used in part (a). □

The preceding lemma says that $\{R(f\theta)_* C_{X_\bullet/\mathcal{Y}_n}\}_n$ is a consistent system, as defined in B.4, in [BO78]. Then, by corollary B.9 in [BO78], it follows that

$$R f_{X_A/\mathcal{Y},crys*}(\mathcal{O}_{X_A/\mathcal{Y},crys}) = R \varprojlim R f_{X_A/\mathcal{Y}_n,crys,*}(\mathcal{O}_{X_A/\mathcal{Y}_n,crys})$$

is bounded above and has finitely generated cohomologies. Thus, we have the following:

Theorem 4.5.1. *$R f_{X_A/\mathcal{Y},crys*}(\mathcal{O}_{X_A/\mathcal{Y},crys})$ is a perfect complex of isocoherent sheaves on $\mathcal{Y}$. Moreover, the isocoherent cohomology sheaf*

$$R^m f_{X_A/\mathcal{Y},crys*}(\mathcal{O}_{X_A/\mathcal{Y},crys})$$

admits a Frobenius structure for each m.

Proof. The first assertion follows from the above paragraph and theorem 1.16 in [Sh08]. The Frobenius structure is given by 2.24 in [HK94], since f is of Cartier type. Indeed, recall that f has a local chart $(P_{X_A} \to M_A, Q_Y \to N, Q \to P)$ given by $Q = \mathbb{N}$, $P = \mathbb{N}^r$, and $Q \to P$ the diagonal map. $\square$

Now let us consider the following commutative diagram, where all squares are cartesian:

$$\begin{tikzcd}
(X_0, M_0) \arrow[r, "f_s"] \arrow[d] & s^{\times} \arrow[r] \arrow[d] & \mathrm{Spf}\ W^{\times} \arrow[d, "\varphi"] \\
(X_A, M_A) \arrow[r, "f"'] & (Y, N) \arrow[r, "\iota"'] & (\mathcal{Y}, \mathcal{N})
\end{tikzcd} \tag{4.11}$$

By theorem 1.19 in [Sh08], we have the following base change property.

Theorem 4.5.2. *In diagram (4.11), there is a quasi-isomorphism*

$$L\varphi^* Rf_{X_A/\mathcal{Y},crys*}(\mathcal{O}_{X_A/\mathcal{Y},crys}) \xrightarrow{\sim} Rf_{s,X_0/W,crys,*}(\mathcal{O}_{X_0/W,crys}).$$

Note that $Rf_{s,X_0/W,crys,*}(\mathcal{O}_{X_0/W,crys})$ is a perfect K-complex that gives the cohomology

$$H^i_{log-crys}((X_0, M_0)/W^{\times}) \otimes K.$$

Relative Log Convergent Cohomology

Following [Sh08], we study the relative log convergent cohomology sheaves there defined. Again, we work only with the trivial isocrystal $\mathcal{O}_{X_A/\mathcal{Y},conv}$, on the log convergent site, and denote the sheaves of relative cohomology by $Rf_{X_A/\mathcal{Y},conv*}(\mathcal{O}_{X_A/\mathcal{Y},conv})$.

Recall that there is a canonical functor (see 2.34 in [Sh08]) from the category of isocrystals on the relative log convergent site to that on log crystalline site

$$\Phi : I_{conv}((X_A/\mathcal{Y})^{log}_{conv}) \longrightarrow I_{crys}((X_A/\mathcal{Y})^{log}_{crys})$$

sending locally free isocrystals on $(X_A/\mathcal{Y})^{log}_{conv}$ to locally free isocrystals on $(X_A/\mathcal{Y})^{log}_{crys}$. In particular, $\Phi(\mathcal{O}_{X_A/\mathcal{Y},conv}) = \mathcal{O}_{X_A/\mathcal{Y},crys}$.

Now let us go back to the situation in diagram (4.10). Let $]X_\bullet[^{log}_{\mathcal{P}_\bullet}$ be the log tube of the closed immersion $i_\bullet$, and $\widehat{\mathcal{P}_\bullet}$ the completion of $\mathcal{P}_\bullet$ along $X_\bullet$. Then, as in [CT12], we have a specialization map

$$\mathrm{sp} :]X_\bullet[^{log}_{\mathcal{P}_\bullet} \to \widehat{\mathcal{P}_\bullet}.$$

Moreover, if we denote by $\Omega^\bullet_{]X_\bullet[^{log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle$ the logarithmic De Rham complex of the simplicial rigid analytic space $]X_\bullet[^{log}_{\mathcal{P}_\bullet}$ over the generic fiber $\mathcal{Y}_K$ of $\mathcal{Y}$, then by corollary 2.34 in [Sh08], we have

$$Rf_{X_A/\mathcal{Y},conv*}(\mathcal{O}_{X_A/\mathcal{Y},conv}) \cong R(f\theta)_*\mathrm{sp}_*\Omega^\bullet_{]X_\bullet[^{log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle.$$

Now by using the remarks of page 31 in [Sh08] and passing to the projective limit, we have a canonical morphism of complexes

$$\mathrm{sp}_*\Omega^\bullet_{]X_\bullet[^{log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle \longrightarrow \varprojlim_n C_{X_\bullet/\mathcal{Y}_n}, \tag{4.12}$$

which by theorem 2.36 in [Sh08] gives the following:

Theorem 4.5.3. *The canonical morphism (4.12) induces an isomorphism*

$$R^m f_{X_A/\mathcal{Y},conv*}(\mathcal{O}_{X_A/\mathcal{Y},conv}) \cong R^m f_{X_A/\mathcal{Y},crys*}(\mathcal{O}_{X_A/\mathcal{Y},crys})$$

of isocoherent sheaves on $\mathcal{Y}$.

In particular, by theorem 4.5.1 this allows to prove that

$$Rf_{X_A/\mathcal{Y},conv*}(\mathcal{O}_{X_A/\mathcal{Y},conv})$$

is a perfect complex of isocoherent sheaves, and a base change theorem:

Theorem 4.5.4. *With the same notation in diagram (4.11), there is a natural isomorphism*

$$L\varphi^* Rf_{X_A/\mathcal{Y},conv*}(\mathcal{O}_{X_A/\mathcal{Y},conv}) \cong Rf_{sX_0/W,conv*}(\mathcal{O}_{X/\mathcal{Y},conv}).$$

The complex $Rf_{sX_0/W,conv*}(\mathcal{O}_{X/\mathcal{Y},conv})$ gives the cohomology

$$H^i_{log-conv}((X_0, M_0)/W^\times).$$

Relative Log Analytic Cohomology

Now we study the sheaves of relative log analytic cohomology. Note that g in diagram (4.10) induces a morphism $g_K^{ex} :]X_\bullet[^{log}_{\mathcal{P}_\bullet} \to \mathcal{Y}_K$. Then, the log analytic cohomolgy sheaves of $(X_A, M_A)/(Y, N)$ with respect to $(\mathcal{Y}, \mathcal{N})$ can be computed by (see 4.1 in [Sh08])

$$R^m f_{X_A/\mathcal{Y},an*}(\mathcal{O}_{X_A/\mathcal{Y},an}) = R^m g^{ex}_{K*}\Omega^\bullet_{]X_\bullet[^{log}_{\mathcal{P}_\bullet}/\mathcal{Y}_K}\langle \mathcal{M}_\bullet/\mathcal{N}\rangle.$$

Then, by applying theorem 4.6 in [Sh08], we have the following comparison theorem

Theorem 4.5.5. *Let sp be the specialization map $\mathcal{Y}_K \to \mathcal{Y}$. Then for each m, $R^m f_{X_A/\mathcal{Y},an*}(\mathcal{O}_{X_A/\mathcal{Y},an})$ is a coherent sheaf on $\mathcal{Y}_K$, and there is an isomorphism*

$$sp_* R^m f_{X_A/\mathcal{Y},an*}(\mathcal{O}_{X_A/\mathcal{Y},an}) \cong R^m f_{X_A/\mathcal{Y},conv*}(\mathcal{O}_{X_A/\mathcal{Y},conv}).$$

4.6 Reduction to the case of a family over a curve

Now that we have relative cohomology sheaves defined for the family over Y, we want to restrict those sheaves to a smaller family. Namely, a family over a curve, in order to be in the same situation as in [CT12].

Let us first construct the curve that we shall use. As stated at the beginning of the preceding section, A_0 is a smooth W-algebra. Let $\widetilde{Y} = \text{Spec } A_0$ and $S = \text{Spec } W$. Since $Y \to \widetilde{Y}$ is a closed immersion, the image $\hat{s}$ of s inside $\widetilde{Y}$ is a closed point. Since the natural morphism $\widetilde{Y} \to S$ is smooth, there exists an affine open neighborhood $\widetilde{U}$ of $\hat{s}$ and an étale morphism $\sigma : \widetilde{U} \to \mathbb{A}^d_W$ such that $\widetilde{U} \to S$ factorizes in the following way:

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\sigma"] \arrow[d] & \mathbb{A}^d_W \arrow[dl] \\ S & \end{tikzcd}$$

Let us recall this construction. There exists an open affine subset $\widetilde{U} = \mathrm{Spec}\ (A_0)_g$ of $\widetilde{Y}$ such that the restriction of $\widetilde{Y} \to S$ is standard smooth, i.e., locally defined by a morphism of rings of the type

$$R \to R[x_1, ..., x_n]/(f_1, ..., f_c) =: S,$$

where the polynomial $\det(\partial f_i/\partial x_j)_{1 \le i,j \le c}$ is invertible in S.

Moreover, we may assume (using the fact that the reduction modulo p is smooth over $k[t]$) that we can write

$$(A_0)_g = W[x_1, ..., x_r, t]/(f_1, ..., f_c),$$

where the polynomial

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_c}{\partial x_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_1}{\partial x_c} & \cdots & \frac{\partial f_c}{\partial x_c} \end{bmatrix}$$

is invertible in $(A_0)_g$. Then, the morphism $W[x_{c+1}, ..., x_r, t] \to (A_0)_g$ is étale, and with $d = r + 1 - c$ we get the desired factorization.

Using this description, it is clear how to construct a smooth curve C_W inside $\widetilde{U}$, transversal to $(t = 0)$ and passing through the point $\hat{s}$: by pulling back a curve with these properties inside $\mathbb{A}^d_W$. In particular, its reduction C modulo p is a smooth curve inside Y, transversal to $(t = 0)$ and passing throught the point s.

Let N_C be the log structure on C defined to make the closed immersion $(C, N_C) \to (Y, N)$ exact, and then we have a sequence of exact closed immersions

$$s^\times \to (C, N_C) \to (Y, N).$$

Let $(X_C, M_C) = (X_A, M_A) \times_{(Y,N)} (C, N_C)$. Then, we have the following diagram, where all the squares are cartesian:

$$\begin{tikzcd}
(X_0, M_0) \arrow[r] \arrow[d] & (X_C, M_C) \arrow[r] \arrow[d] & (X_A, M_A) \arrow[d] \\
s^{\times} \arrow[r] & (C, N_C) \arrow[r] & (Y, N)
\end{tikzcd}$$

Note that the family $(X_C, M_C) \to (C, N_C)$ is in the situation studied in [CT12]. We denote by $\mathcal{C}$ the p-adic completion of C_W along the special fiber C. Then $1 \mapsto t$ defines a log structure $\mathcal{N}_{\mathcal{C}}$ on $\mathcal{C}$ and we have the following diagram

$$\begin{tikzcd}
(X_C, M_C) \arrow[r, "f_C"'] \arrow[d] & (C, N) \arrow[r] \arrow[d] & (\mathcal{C}, \mathcal{N}_{\mathcal{C}}) \arrow[d, "\iota"] \\
(X_A, M_A) \arrow[r, "f"] & (Y, N) \arrow[r] & (\mathcal{Y}, \mathcal{N})
\end{tikzcd}$$

Then, by theorem 1.19, corollary 2.38 in [Sh08], we have an isomorphism

$$L\iota^* Rf_{X_A/\mathcal{Y},crys*}(\mathcal{O}_{X_A/\mathcal{Y},crys}) \xrightarrow{\sim} Rf_{C,X_C/\mathcal{C},crys*}(\mathcal{O}_{X_C/\mathcal{C},crys}) \tag{4.13}$$

Now consider the diagram

$$\begin{tikzcd}
(X_0, M_0) \arrow[r] \arrow[d] & s^{\times} \arrow[r] \arrow[d] & \mathrm{Spf}\, W^{\times} \arrow[d, "\psi"] \\
(X_C, M_C) \arrow[r] & (C, N_C) \arrow[r] & (\mathcal{C}, \mathcal{N}_{\mathcal{C}})
\end{tikzcd}$$

where $\iota \circ \psi = \varphi$. Then we have an isomorphism

$$L\psi^* Rf_{C,X_C/\mathcal{C},crys*}(\mathcal{O}_{X_C/\mathcal{C},crys}) \xrightarrow{\sim} Rf_{s,X_0/W,crys*}(\mathcal{O}_{X_0/W,crys}). \tag{4.14}$$

By combining the isomorphisms (4.13) and (4.14), and the fact that $L\psi^* L\iota^* \cong L(\psi^*\iota^*) \cong L((\iota \circ \psi)^*) = L\psi^*$, we get that $Rf_{s,X_0/W,crys*}(\mathcal{O}_{X_0/W,crys})$ can be obtained from the family over Y or over C. In particular, by the main result in [CT12], we get the following Clemens-Schmid type exact sequence:

$$\cdots \to H^m_{rig}(X_0) \to H^m_{log-crys}((X_0, M_0)/W^\times) \otimes K \xrightarrow{N} \tag{4.15}$$

$$H^m_{log-crys}((X_0, M_0)/W^\times) \otimes K(-1) \to H^{m+2}_{X_0,rig}(X_C) \to H^{m+2}_{rig}(X_0) \to \cdots$$

The terms of the form $H^{m+2}_{X_0,rig}(X_C)$ depend a priori on the choice of the curve C, but if we choose a different smooth curve C', by Poincaré duality (theorem 2.4 in [Be97i]), we have isomorphisms

$$H^{m+2}_{X_0,rig}(X_C) \cong H^{m+2}_{X_0,rig}(X_{C'}) \cong H^{2\dim X-m-2}_{c,rig}(X_0)^\vee(-\dim X)$$

$$\cong H^{rig}_{2\dim X_0-m}(X_0)(-\dim X),$$

and we get a Clemens-Schmid type exact sequence that depends only on X and the special fiber X_0 for our starting situation.

Chapter 5

Monodromy Criteria

Recall from the first section of chapter 4 that in the classical situation, for a semistable degeneration of $K3$ surfaces one can determine the behavior of the central fiber by the degree of nilpotency of the monodromy operator N on the limit cohomology. This is done by first proving criteria for N being zero.

In this chapter we shall prove the p-adic analogue of these results. Namely, we shall get in the first section criteria for the monodromy operator (on the log-crystalline cohomology) being zero. Then, in the second section we apply these criteria to the case of combinatorial $K3$ surfaces over finite fields. This shall allow us to get our main theorem and consequently a criterion for good reduction of $K3$ surfaces.

5.1 General Monodromy Criteria

As an application of the p-adic version of the Clemens-Schmid exact sequence, we prove a p-adic version of the Monodromy Criteria (p. 112 in [Mo84]). We start with a situation in which we have an exact sequence of Clemens-Schmid type, as for example the situation in [CT12] or the situation of section 4.3, that we saw that it is reduced to the first one.

Suppose k is a finite field and C a smooth curve over k. We consider a proper and flat morphism

$$f : X \to C,$$

where X is a smooth variety of dimension $n+1$ over k. Moreover, we assume that there exists a k-rational point $s \in C$ such that the fiber of f at s, which we denote by X_s is a NCD. This defines a log structure M on X. We denote by (X_s, M_s) the log scheme with the induced log structure.

Then, the main result of [CT12] states that there is a long exact sequence:

$$\cdots \to H^m_{rig}(X_s) \to H^m_{log-crys}((X_s, M_s)/W^\times) \otimes K \to$$

$$H^m_{log-crys}((X_s, M_s)/W^\times) \otimes K(-1) \to H^{m+2}_{X_s, rig}(X) \to H^{m+2}_{rig}(X_s) \to \cdots$$

We can consider the maps as morphisms of filtered vector spaces, where we give the weight filtration to each of them. Let us make a description of this: denote by $X_1, ..., X_r$ the irreducible components of X_s and assume they are proper and smooth. Define the *codimension p stratum of X_s* as

$$X^{[p]} := \bigsqcup_{i_0 < \cdots < i_p} X_{i_0} \cap \cdots \cap X_{i_p}.$$

For each $a = 0, ..., p+1$, denote by $\delta_a : X^{[p+1]} \to X^{[p]}$ the natural map that restricted to each component is the inclusion

$$X_{i_0} \cap \cdots \cap X_{i_{p+1}} \hookrightarrow X_{i_0} \cap \cdots \cap X_{i_{a-1}} \cap X_{i_{a+1}} \cap \cdots \cap X_{i_{p+1}}$$

and define

$$\rho_p := (-1)^p \sum_{a=0}^{p} (-1)^a \delta_a^*, \tag{5.1}$$

where δ_a^* is the morphism of De Rham-Witt complexes $W_n\Omega^\bullet_{X^{[p]}} \to W_n\Omega^\bullet_{X^{[p+1]}}$ induced by δ_a, where we identify $W_n\Omega^\bullet_{X^{[p]}}$ with its direct image in the étale site of X_s.

This gives a double complex

$$0 \longrightarrow W_n\Omega^\bullet_{X^{[0]}} \xrightarrow{\rho_0} W_n\Omega^\bullet_{X^{[1]}} \xrightarrow{\rho_1} W_n\Omega^\bullet_{X^{[2]}} \xrightarrow{\rho_2} \cdots \tag{5.2}$$

and by taking projective limit, we get the double complex

$$0 \longrightarrow W\Omega^\bullet_{X^{[0]}} \xrightarrow{\rho_0} W\Omega^\bullet_{X^{[1]}} \xrightarrow{\rho_1} W\Omega^\bullet_{X^{[2]}} \xrightarrow{\rho_2} \cdots \tag{5.3}$$

This allows to define a spectral sequence with

$$E_1^{p,q} = H^q_{rig}(X^{[p]}) \tag{5.4}$$

with $d_1^{p,q}$ induced by ρ_p.

Theorem 5.1.1. *The spectral sequence (5.4) degenerates at E_2 and converges to $H^*_{rig}(X_s)$.*

Proof. Since $X^{[p]}$ is smooth and proper, then $H^q_{rig}(X^{[p]})$ is pure of weight of q. Since $E_2^{p,q}$ is a sub quotient of this, we have that

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

has to be the zero morphism, which proves the degeneracy. To prove that it converges to $H^*_{rig}(X_s)$, it is enough to notice that the simple complex associated to (5.3) gives this cohomology. This is proposition 1.8 and theorem 3.6 of [Ch99].

□

Mokrane defined the following spectral sequence in [Mk93][1]:

$$E_1^{-k,i+k} = \bigoplus_{j\geq 0, j\geq -k} H^{i-2j-k}_{crys}(X^{[2j+k]}/W)(-j-k) \Longrightarrow H^i_{log-crys}((X_s, M_s)/W^\times)$$

Then, theorem 3.32 in [Mk93] asserts that if X_s is projective, this sequence degenerates at E_2 modulo torsion, and it gives the weight filtration on log-crystalline cohomology.

The weight filtration on rigid cohomology (given by the Frobenius operator) is induced by the spectral sequence (5.4). Now we list some properties of this filtration, denoted by $W_\bullet$, and its respective graded modules $Gr_\bullet$ on $H^m_{log-crys} := H^m_{log-crys}((X_s, M_s)/W^\times) \otimes K$ and $H^m_{rig} := H^m_{rig}(X_s)$, which are just a consequence of the previous remarks and theorem 5.1.1.

Proposition 5.1.1. *(i) N^k induces an isomorphism of vector spaces*

$$Gr_{m+k} H^m_{log-crys} \xrightarrow{\sim} Gr_{m-k} H^m_{log-crys}$$

[1]Note that in Mokrane's article, there is a shift in the indices of the strata.

for all $k \geq 0$.

(ii) *For* $k \leq m$, *we have a decomposition*

$$Gr_k(H^m_{log-crys}) = \bigoplus_{a=0}^{[k/2]} Gr_{k-2a}(\mathcal{K}_m),$$

where $\mathcal{K}_m = \ker N \subset H^m_{log-crys}$, *and the filtration on* $\mathcal{K}_m$ *is induced by the one on* $H^m_{log-crys}$.

(iii) $Gr_0(H^m_{rig}) = H^m(|\Gamma|)$, *where* Γ *is the dual graph associated to* X_s.

(iv) $Gr_k(H^m_{rig}) = E_2^{m-k,k}$.

We also need the following, which is an immediate corollary of the Clemens-Schmid exact sequence.

Proposition 5.1.2. *For all* $k < m$, $W_k(H^m_{rig}) \cong W_k(\mathcal{K}_m)$.

Proof. It is enough to note that since $H^m_{X_s,rig}(X)$ has weights $> m-1$, when restricting the Clemens-Schmid sequence to the W_k-parts we get an exact sequence:

$$0 \to W_k(H^m_{rig}) \to W_k(\mathcal{K}_m) \to 0$$

for $k < m$. $\square$

With the two previous propositions in hand, we can prove the following monodromy criteria:

Theorem 5.1.2. *Denote* $H^i_{log-crys} := H^i_{log-crys}((X_s, M_s)/W^\times) \otimes K$. *Let* $h^k(|\Gamma|) = \dim H^k(|\Gamma|)$, $b_k(X_s) = \dim H^k_{log-crys}$, $h^k(X^{[j]}) = \dim H^k_{rig}(X^{[j]})$ *and* $\Phi = \dim Gr_1 H^1_{rig}$. *Then, we have the following:*

(i) $N = 0$ *on* $H^1_{log-crys}$ *if and only if* $h^1(|\Gamma|) = 0$ *if and only if* $b_1(X_s) = \Phi$.

(ii) $N^2 = 0$ *on* $H^2_{log-crys}$ *if and only if* $h^2(|\Gamma|) = 0$.

(iii) $N = 0$ *on* $H^2_{log-crys}$ *if and only if* $h^2(|\Gamma|) = 0$ *and* $\Phi = h^1(X^{[0]}) - h^1(X^{[1]})$

Proof. (i) By the final remark in [Ch99], we have an exact sequence

$$0 \to H^1_{rig} \to H^1_{log-crys} \xrightarrow{N} H^1_{log-crys}.$$

In particular, $\ker N \cong H^1_{rig}$. Then, by part (i) of proposition 5.1.1, we have $Gr_2 H^1_{log-crys} \cong Gr_0 H^1_{log-crys}$, and by part (ii), we have $Gr_0 H^1_{log-crys} \cong Gr_0(\mathcal{K}_1) = Gr_0(H^1_{rig})$, and by part (iii), we conclude that $Gr_2 H^1_{log-crys} \cong H^1(|\Gamma|)$.

Similarly, by part (ii) of proposition 5.1.1, we have $Gr_1 H^1_{log-crys} \cong Gr_1(\mathcal{K}_1) = Gr_1 H^1_{rig}$.

First suppose $N = 0$. Then, $Gr_2 H^1_{log-crys} = 0 = Gr_0 H^1_{log-crys}$, since the first isomorphism is induced by N. Then it follows that $h^1(|\Gamma|) = 0$ and $b_1(X_s) = \Phi$.

Now suppose that $h^1(|\Gamma|) = 0$. Then, $Gr_0 H^1_{log-crys} \cong Gr_0(H^1_{rig}) = 0$. This implies that $Gr_1 H^1_{log-crys} = H^1_{log-crys}$, but $Gr_1 H^1_{log-crys} = Gr_1 \mathcal{K}_1$, hence $Gr_1 \mathcal{K}_1 = H^1_{log-crys}$. By part (ii) of proposition 5.1.1, we also have $Gr_0 \mathcal{K}_1 \cong Gr_0 H^1_{log-crys} = 0$ and

$$Gr_2 \mathcal{K}_1 \oplus Gr_0 \mathcal{K}_1 = Gr_2 \mathcal{K}_1 \cong Gr_2 H^1_{log-crys} = 0,$$

hence $\mathcal{K}_1 = Gr_1 \mathcal{K}_1 = H^1_{log-crys}$, which proves that $N = 0$.

Finally, note that if $b_1(X_s) = \Phi$, then $Gr_1 H^1_{log-crys} = H^1_{log-crys}$, and this implies that $h^1(|\Gamma|) = 0$.

(ii) For the proof of this and next part, we note that the Clemens-Schmid sequence for even indices can be seen as two exact sequences (since $N = 0$ on $H^0_{log-crys}$):

$$0 \to H^0_{rig} \to H^0_{log-crys} \to 0$$

$$0 \to H^0_{log-crys} \to H^2_{X_s,rig}(X) \to H^2_{rig} \to H^2_{log-crys} \xrightarrow{N} H^2_{log-crys} \to \cdots$$

By part (ii) of proposition 5.1.1, we have that $Gr_0 H^2_{log-crys} \cong Gr_0 \mathcal{K}_2$, and by proposition 5.1.2, this is isomorphic to $Gr_0 H^2_{rig} \cong H^2(|\Gamma|)$.

Suppose that $N^2 = 0$ on $H^2_{log-crys}$. Then, by part (i) of proposition 5.1.1, we have $Gr_4 H^2_{log-crys} \cong Gr_0 H^2_{log-crys} = 0$, and this gives that $h^2(|\Gamma|) = 0$.

Conversely, suppose that $h^2(|\Gamma|) = 0$. Then, $\dim Gr_0 H^2_{rig} = 0$. Note that N^2 takes W_0 to $W_{-4} = 0$, W_1 to $W_{-3} = 0$, W_2 to $W_{-2} = 0$, W_3 to $W_{-1} = 0$ and W_4 to

$W_0 = Gr_0 H^2_{rig} = 0$. Thus, $N^2 = 0$.

(iii) By part (ii) of proposition 5.1.1, we have that $Gr_1 H^2_{log-crys} \cong Gr_1 \mathcal{K}_2$, and by proposition 5.1.2, this is isomorphic to $Gr_1 H^2_{rig} = E_2^{1,1} = \ker d_1^{1,1} / \operatorname{Im} d_1^{0,1}$.
Note that $d_1^{1,1} : H^1_{rig}(X^{[1]}) \to H^2_{rig}(X^{[1]})$ is the zero map (since $H^2_{rig}(X^{[1]}) = 0$). Then, we conclude that

$$\dim Gr_1 H^2_{log-crys} = h^1(X^{[1]}) - \dim \operatorname{Im}\,(H^1_{rig}(X^{[0]}) \to H^1_{rig}(X^{[1]}))$$

$$= h^1(X^{[1]}) - (h^1(X^{[0]}) - \dim \ker(H^1_{rig}(X^{[0]}) \to H^1_{rig}(X^{[1]})))$$

$$= \Phi - h^1(X^{[0]}) + h^1(X^{[1]}).$$

Now suppose that $N = 0$. Then, $N^2 = 0$ and by the preceding part, we have $h^2(|\Gamma|) = 0$. Moreover, N induces an isomorphism

$$Gr_3(H^2_{log-crys}) \xrightarrow{\sim} Gr_1(H^2_{log-crys}),$$

hence $Gr_1 H^2_{log-crys} = 0$ and $\Phi - h^1(X^{[0]}) + h^1(X^{[1]}) = 0$.
Conversely, suppose that $h^2(|\Gamma|) = 0$ and $\Phi - h^1(X^{[0]}) + h^1(X^{[1]}) = \dim Gr_1 H^2_{log-crys} = 0$ and let us prove that $\mathcal{K}_2 = H^2_{log-crys}$ (hence $N = 0$). First note that $Gr_0 H^2_{log-crys} = 0$, since $h^2(|\Gamma|) = 0$, i.e., $W_0 = 0$. But since $Gr_1 H^2_{log-crys} = 0$, then $W_1 = 0$. By part (ii) of proposition 5.1.1, we have $Gr_3 H^2_{log-crys} = 0$, hence $W_3 = W_2$. By the same argument, $Gr_0 H^2_{log-crys} \cong Gr_4 H^2_{log-crys}$, hence $W_4 = W_3 = W_2 = H^2_{log-crys}$. This gives $Gr_2 H^2_{log-crys} = H^2_{log-crys}$. By part (ii) of proposition 5.1.1, we get

$$Gr_2 H^2_{log-crys} = Gr_2 \mathcal{K}_2 \oplus Gr_0 \mathcal{K}_2 = Gr_2 \mathcal{K}_2 = \mathcal{K}_2,$$

which concludes the proof.

□

5.2 The Main Theorem

In this section we assume $p > 3$. Let K be a finite extension of $\mathbb{Q}_p$ and denote by O_K its ring of integers, π a uniformizer of O_K and k its residue field. We consider a smooth, projective $K3$ surface $X_K \to \text{Spec } K$ with a semi-stable model $X \to \text{Spec } O_K$, i.e., X is a proper scheme over O_K, étale locally étale over a scheme of the form $\text{Spec }(O_K[x_1, ..., x_n]/(x_1 \cdots x_r - \pi))$. Let $X_s := X \otimes_{O_K} k$ be the special fiber of X and assume it is a combinatorial $K3$ surface. In particular, we may assume that we are in one of the following cases:

I) X_s is a smooth $K3$ surface over k

II) $X_s = X_0 \cup X_1 \cup \cdots \cup X_{j+1}$ is a chain of smooth surfaces, with X_0, X_{j+1} rational and the others are elliptic ruled and two double curves on each of them are rulings.

III) $X_s = X_0 \cup X_1 \cup \cdots \cup X_{j+1}$, with every X_i a rational surface, and the double curves on X_i are rational and form a cycle on X_i. The dual graph of X is a triangulation of the sphere S^2.

We shall refer to each of these as surface of type I, II and III, respectively.

Remark 5.2.1. The definition of a combinatorial $K3$ surface (definition 3.3.5)requires for the cases II) and III) that the geometric special fiber $X_{\overline{s}}$ has a decomposition of those types and not necessarily X_s, but this implies that there exists a finite extension k' of k such that the base change $X_{k'} = X_s \otimes_k k'$ has such decomposition. Since k' is again a finite field, we may assume that it is X_s the one that admits such decomposition.

To study surfaces of type II we shall use the following:

Lemma 5.2.1. *Let Y be a smooth, proper, rational surface over a field. Then, $H^1_{rig}(Y) = 0$.*

Proof. First note that since Y is smooth and proper over a field, then Y is necessarily projective (this is remark 3.5 in chapter 9 of [Liu]). Then, we use Castelnuovo-Zariski's criterion in characteristic p as stated in theorem 4.6 of [Li13] to get that the first ℓ-adic étale cohomology group is trivial. Since Y is smooth and proper, we conclude that the dimension of the first rigid cohomology group is also 0, by the Weil cohomology formalism. □

Recall from [Na00] that the special fiber X_s can be endowed with a log structure M_s in such a way that we have a log smooth morphism $(X_s, M_s) \to (\mathrm{Spec}\ k, \mathbb{N}^m)$, where m is the number of connected components of the singular locus of X_s and the log structure is defined by $e_i \mapsto 0$, where e_i denotes the ith canonical generator of $\mathbb{N}^m$.

Let us make an explicit description of M_s. In general, suppose that Y is a normal crossing variety and denote by Y_{sing} the singular locus. Denote by $Y_1, ..., Y_m$ the connected components. For each $i = 1, ..., m$, we can endow Spec $(k[x_0, ..., x_n]/(x_0 \cdots x_r))$ with a log structure given by as follows:

$$\mathbb{N}^{m+r} = \mathbb{N}^{i-1} \oplus \mathbb{N}^{r+1} \oplus \mathbb{N}^{m-i} \to k[x_0, ..., x_n]/(x_0 \cdots x_r)$$

$$e_i \mapsto \begin{cases} 0 & \text{if } e_i \in \mathbb{N}^{i-1} \\ x_{i-1} & \text{if } e_i \in \mathbb{N}^{r+1} \\ 0 & \text{if } e_i \in \mathbb{N}^{m-i} \end{cases}$$

Then,

1. If x is a smooth point of Y, étale locally on a neighbourhood of x, the log structure is the pull-back of the log structure of the log-point $(\mathrm{Spec}\ k, \mathbb{N}^m)$

2. If $x \in Y_i$, étale locally on a neighbourhood of x, the log structure is the pull-back of the log structure defined above.

Since X_s is in particular a normal crossing variety over k, we can endow X_s with this log structure and we denote it by M_s. Note that this is not the usual log structure defined for example in [Ka89], which we denote here by M'_s. As it is stated in [Na00], the relationship between them is

$$(X_s, M'_s) = (X_s, M_s) \times_{(\mathrm{Spec}\ k, \mathbb{N}^m)} (\mathrm{Spec}\ k, \mathbb{N}),$$

where the morphism of log schemes $(\mathrm{Spec}\ k, \mathbb{N}) \to (\mathrm{Spec}\ k, \mathbb{N}^m)$ is defined by $s : \mathbb{N}^m \to \mathbb{N}$ the sum of the components. Moreover, the sheaves of relative log differentials $\omega^\bullet_{(X_s, M_s)/(\mathrm{Spec}\ k, \mathbb{N}^m)}$ and $\omega^\bullet_{(X_s, M'_s)/(\mathrm{Spec}\ k, \mathbb{N})}$ coincide, and there is also a canonical isomorphism

$$H^i_{log-crys}((X_s, M_s)/(W, \mathbb{N}^m)) \cong H^i_{log-crys}((X_s, M'_s)/(W, \mathbb{N})),$$

as stated and proved in the appendix of [Na00].

Assume for the moment that X_s is either of type I), type III) or type II) such that the double curve is ordinary. Then, by corollary 5.4 and proposition 5.9 in [Na00], there exists a semistable family $X^{\log}$ over Spec $k[[t]]^{\log}$ such that its special fiber is precisely (X_s, M_s). Then we have the following diagram, with cartesian squares:

$$\begin{tikzcd}
(X_s, M'_s) \arrow[r] \arrow[d] & (X_s, M_s) \arrow[r] \arrow[d] & X^{\log} \arrow[d] \\
(\text{Spec } k, \mathbb{N}) \arrow[r] & (\text{Spec } k, \mathbb{N}^m) \arrow[r] & \text{Spec } k[[t]]^{\log}
\end{tikzcd} \tag{5.5}$$

Let $\widetilde{X}$ be the underlying scheme of $X^{\log}$. Then we can apply the same technique as in section 4.6 and get a smooth curve C over k, and a regular scheme X_C with a proper, flat morphism $X_C \to C$ such that there exists a k-rational point $s \in C$ such that the fiber of f at s is precisely X_s. Then, we have the following:

Theorem 5.2.1. *(a) X_s is of type I if and only if $N = 0$ on $H^2_{log-crys}$.*

(b) X_s is of type II if and only if $N \neq 0$ and $N^2 = 0$ on $H^2_{log-crys}$.

(c) X_s is of type III if and only if $N^2 \neq 0$ on $H^2_{log-crys}$.

Proof. We shall prove that if $N = 0$ on $H^2_{log-crys}$, then X_s is neccesarily of type I; if $N \neq 0$ and $N^2 = 0$, then X_s is necessarily of type II; and if $N^2 \neq 0$, then X_s is necessarily of type III. This shall prove the equivalence, since we know that we can be only in one of these three cases.

First assume that X_s is of type I. Then, $X^{[0]} = X_s$, $X^{[1]} = \varnothing$ and the dual graph Γ is only one point. In this case, the spectral sequence has the form

$$E_\infty^{p,q} = E_1^{p,q} = H^q_{rig}(X^{[p]}) = \begin{cases} 0 & \text{if } p \geq 1 \\ H^q_{rig}(X_s) & \text{if } p = 0 \end{cases}$$

and this gives immediately that $\Phi = \dim Gr_1 H^1_{rig} = \dim E_2^{0,1} = 0$. Since $H^1_{rig}(X_s) = H^1_{rig}(X^{[1]}) = 0$, and $h^2(|\Gamma|) = 0$, we conclude that $N = 0$, by theorem 5.1.2 (iii).

Now assume that X_s is of type II (use the same notation as in the beginning of the section). In this case, it is clear that the dual graph is homeomorphic to $[0,1]$. In particular, $h^2(|\Gamma|) = 0$ and $N^2 = 0$ by theorem 5.1.2 (ii). By definition of the type II, $X^{[1]}$ is the disjoint union of $j+1$ elliptic curves, hence $h^1(X^{[1]}) = 2j+2$. Since X_0 and X_{j+1} are rational surfaces, by lemma 5.2.1, we have

$$h^1(X^{[0]}) = \sum_{i=1}^{j} h^1(X_i),$$

but the X_i's are ruled, with the double curves rulings. Then, $h^1(X^{[0]}) = 2j$ and we get $h^1(X^{[0]}) - h^1(X^{[1]}) = -2$, but Φ cannot be negative, hence

$$\Phi \neq h^1(X^{[0]}) - h^1(X^{[1]})$$

and $N \neq 0$. Finally, assume that X_s is of type III. In this case, $h^2(|\Gamma|) = h^2(S^2) = 1 \neq 0$, hence $N^2 \neq 0$.

The only remaining case is when X_s is of type II such that the double curve is not ordinary, i.e., supersingular. In this case, by corollary 6.9 of [Na00], the geometric special fiber $X_{\overline{s}}$ is the special fiber of a projective semistable family $\widetilde{X}$ over Spec $\overline{k}[[t]]$. Now we use the same approximation argument from section 4.4, and we get the following cartesian diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[d] & X_A \arrow[d] \\
\operatorname{Spec} \overline{k}[[t]] \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

where A is a smooth $\overline{k}[t]$-algebra. Locally, X_A is defined by a finite number of polynomials in $\overline{k}[t]$, and in particular they contain a finite number of elements of $\overline{k}$. Then by proposition 2.4.2, there exists a finite extension k' of k and a $k'[t]$-algebra A' over which we can define $X_{A'}$ to have a cartesian diagram

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & X_{A'} \arrow[d, "f'"] \\ \text{Spec } A \arrow[r] & \text{Spec } A' \end{tikzcd}$$

The composition $A' \to A \to \overline{k}[[t]] \to \overline{k}$ defines a closed point x in Spec A'. Then, the fiber of f' at x, denoted by X_x, satisfies

$$X_x \otimes_{k'} \overline{k} \cong X_s \otimes_k \overline{k} = X_{\overline{s}}$$

Then, there exists a finite extension k'' of k' such that

$$X_x \otimes_{k'} k'' \cong X_s \otimes_k k'' =: X_s''.$$

Since $(X_s \otimes_k k'') \otimes_{k''} \overline{k} = X_s'' \otimes_{k''} \overline{k}$, we get that X_s is of the same type (I, II or III) as X_s''. Moreover, if we denote by K''/K the extension corresponding to k''/k, then the degree of nilpotency on $H^2_{log-crys}$ and $H^2_{log-crys} \otimes_K K''$ is preserved, since any extension of fields is faithfully flat. This completes the proof for all the cases.

□

Finally we prove the monodromy criterion for the good reduction of $K3$ surfaces that we stated in the introduction, and we can get as well the main result in [Pe14].

Corollary 5.2.1. *Let $p > 3$ and K a finite extension of $\mathbb{Q}_p$. Let X_K be a smooth, projective $K3$ surface over K, that admits a semistable model over O_K. Then, X_K has good reduction if and only if the monodromy operator N on $H^2_{DR}(X_K)$ is zero.*

Proof. Since $p > 3$, by proposition 3.3.4, X_K has a minimal semistable X model over O_K. In particular, the special X_s is a combinatorial $K3$ surface over a finite field k. By the preceding theorem, X_s is smooth if and only the monodromy operator on its log-crystalline cohomology is zero. But since X_K has a semistable model, this is equivalent to have trivial monodromy on $H^2_{DR}(X_K)$, which is the desired result. □

Corollary 5.2.2 (Pérez Buendía)**.** *Let $p > 3$ and K a finite extension of $\mathbb{Q}_p$. Let X_K be a smooth, projective $K3$ surface over K, that admits a semistable model over O_K. Then, X_K has good reduction if and only if the monodromy operator N_{st} on $D_{st}(H^2_{ét}(X_{\overline{K}}, \mathbb{Q}_p))$ is zero.*

Proof. The proof is almost identical to that of corollary 5.2.1, but now we need to use the comparison isomorphism (theorem 0.2 in [Ts99]) in order to get the same result for the monodromy operator N_{st}. □

Remark 5.2.2. The comparison isomorphism from [Ts99] used to prove corollary 5.2.2 says that our main result is actually equivalent to the main result in [Pe14]. The proofs, however, are very different since the proof made in [Pe14] uses p-adic Hodge Theory and, moreover, it relies on a trascendental argument. Namely, one constructs a semistable degeneration of complex $K3$ surfaces, as defined in section 4.1, which preserves the type of the special fiber and the degree of nilpotency of the monodromy operator. Then, one uses the classical monodromy criteria (theorem 4.1.5) to get the desired result. On the other hand, our proof relies only on p-adic methods and it is largely inspired by the proof of theorem 4.1.5.

Chapter 6

The case of Enriques Surfaces

Once that we have proven our main theorem, we shall try to get a similar result for Enriques surfaces. Namely, we would like to get a good reduction crieterion for semistable Enriques surfaces. In this chapter we describe the techniques that can be used to get that result.

First let us recall some facts about the previous works, including ours. Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics, K its fraction field and k the residue field, which we assume to be perfect. Let $W := W(k)$ denote the ring of Witt-vectors with coefficients in k, seen as a subring of $\mathcal{V}$ and let K_0 denote its fraction field.

For a proper variety X over $\mathcal{V}$ with semistable reduction and special fiber X_k, via the theory of log schemes and the work of Hyodo-Kato in [HK94], one defines a monodromy operator on the de Rham comology groups of its generic fiber X_K. This action has been used to give criteria for good reduction. In the case of abelian varieties, this has been enough. But in a more general situation, it has been proved that such an action is not enough to detect "good reducibility". In fact, even in the case of curves one has to study the unipotent fundamental group (whose abelianization is related to the first cohomology group).

Since $K3$ surfaces are simply connected, the first fundamental group is trivial and the only relevant cohomology group is the second. By the Hurewicz theorem, we have that the second homotopy group π_2 is isomorphic to the second homology group. If we want to state this in terms of cohomology, their second rational homotopy group is associated by duality to the second cohomology group, and in fact using this second cohomology group we can get the criterion.

Moreover, we want to give another hint of the following philosophy: the criteria for good reduction should involve the unipotent fundamental group and the other higher rational homotopy groups. Note that this is exactly the setting of the rational homotopy theory of Sullivan and Morgan. Here we want to focus on the case of the Enriques surfaces, i.e., surfaces with irregularity 0 and such that the square of its canonical divisor is trivial. Moreover, we assume that the characteristic of the residue field is strictly larger than 3. For such a surface, the first homotopy group π_1 is torsion: indeed, it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We would like to get a result similar to that of curves (as in [AIK13]). Note that we cannot study the action on the first homotopy group or the unipotent completion, which is zero for Enriques surfaces. We study the second homotopy group, which does not coincide with the second cohomology group (as it happens in the case of $K3$ surfaces) since they are not simply connected. We need to study the second homotopy group for Enriques surfaces, and the monodromy action on it should give the desired criterion.

6.1 Universal coverings of Enriques and Rational schemes in characteristic p

In this section we consider a (proper) log semistable Enriques surface X defined over a DVR, such that the generic fiber is a smooth proper Enriques surface and the special fiber is a simple normal crossing log Enriques surface (see Nakkajima [Na00]). In this case $\Lambda^2\omega_X$ is invertible and satisfies $(\Lambda^2\omega_X)^{\otimes 2} = \mathcal{O}_X$. Then to this object is associated an étale covering 2:1 :

$$Y \to X,$$

where Y is a proper semistable $K3$ surface. Moreover, its generic fiber is a proper smooth $K3$ surface (classical $K3$ cover of Enriques surfaces). We can see then the covering $Y_K \to X_K$ as the universal covering of X_K (even if it is finite étale, this is one of the few cases where an algebraic variety has a universal covering in algebraic terms). Since it is a covering, we have that $\pi_i(X_K) = \pi_i(Y_K)$ for $i \geq 2$ (without a precise meaning, for the moment). Hence the second homotopy group π_2 of our Enriques surface is the π_2 of the $K3$ surface, which is linked to its second cohomology group.

Once we have done this step, we need to know the shape of the étale universal covering

Y. Recall that we are assuming that the special fiber of the semistable model is one of those explained by Nakkajima in [Na00], and that just as the case of $K3$ surfaces, it is one of three types, called type I, II and III. Moreover, our map is étale even over the special fiber.

6.2 Enriques Surfaces and Monodromy

Let us give the following definition.

Definition 6.2.1. A semistable surface X is called semistable Enriques if it is a regular, projective surface, such that the generic fiber is a smooth Enriques surface and its special fiber X_k is one of the following:

I) X_k is a smooth Enriques surface.

II) $X \otimes_k \overline{k} = X_1 \cup X_2 \cup \cdots X_N$ is a chain of smooth surfaces, with X_1 rational and the others are elliptic ruled and two double curves on them are rulings.

III) $X \otimes_k \overline{k} = X_1 \cup X_2 \cup \cdots X_N$, with every X_i smooth and rational, and the double curves on X_i are rational and form a cycle on X_i. The dual graph of $X \otimes_k \overline{k}$ is a triangulation of the real projective plane $\mathbb{P}^2(\mathbb{R})$.

Note that under our definition, if Λ_X denotes the sheaf of log-differentials (with respect the log-structure given by the special fiber), we have

$$(\Lambda_X^2)^{\otimes 2} = \mathcal{O}_X.$$

Indeed, the restriction to the open X_K of $(\Lambda_X^2)^{\otimes 2}$ is zero by definition of Enriques in characteristic 0, while its restriction to the special fiber is trivial. Hence $(\Lambda_X^2)^{\otimes 2} = \mathcal{O}_X$. From this we have a natural étale covering of X associated to $(\Lambda_X^2)^{\otimes 2}$ (see [CD]).

In the generic fiber we have a smooth $K3$ surface, while in the special fiber we have a normal crossing divisor which looks like a $K3$ surface because we have vanishing of cohomology and triviality of (Λ_X^2).

If the special fiber of our Enriques surface is smooth, the étale lifting would not change the smoothness and then the action of the monodromy on π_2 is the action of the monodromy on the second cohomology group of the smooth $K3$ surface, hence it is trivial.

Let us now work the other cases.

Lemma 6.2.1. *Consider an Enriques NCD with simple components in X_k. Then the étale covering respects the components, i.e., the lifting of a rational component is a rational component and the lifting of an elliptic one is an elliptic one.*

Proof. The geometric fundamental group of a rational surface is zero. Indeed, it is true that every proper, normal, rational variety over an algebraically closed field is simply connected: see SGA 1, XI, Cor. 1.2. This is more than what we had at the beginning of this chapter. Hence the covering is just a base change: hence the variety remains rational. In the other cases: one may expect that by means of the étale covering we may change the shape of the combinatorial. But if we have a finite étale map $\varphi : T \to S$, then we have an injection of the cohomology groups $H^1(S) \hookrightarrow H^1(T)$. Then an elliptic surface (whose cohomology is not zero) cannot be transformed into a rational one (whose cohomology is zero). Moreover the étale map does not change the fact that the components are smooth. □

Corollary 6.2.1. *The étale $K3$ covering of any Enriques SNCL is a combinatorial SNCL $K3$ surface, and we have a correspondance on the type.*

We may conclude that by using the étale cover given by the Λ^2_X of the given semistable Enriques surface X, we are reduced to a semistable $K3$ surface Y, whose special fiber is a combinatorial one. In particular, the generic fiber gives a universal covering (since a smooth $K3$ surface is simply connected) of the given Enriques surface. We then have that

$$\pi_2(X_K) = \pi_2(Y_K) = H^2_{dR}(Y_K),$$

since for a smooth $K3$ surface, the second homotopy group coincide with the second cohomology group. For the universal covering, the higher homotopy groups do not change.

Recall that the action of the monodromy on $H^2_{dR}(X_K)$ was zero. Now we look at the action on the fundamental group: we have a natural immersion $H^2_{dR}(X_K) \to H^2_{dR}(Y_K) = \pi_2(Y_K) = \pi_2(X_K)$, and moreover the étale map $Y \to X$ gives a morphism in cohomology compatible with monodromy (even if it is not a priori). Thus, the final result should be:

Conjecture 6.2.1. *A semistable Enriques surface is smooth if and only if the action of the monodromy on second homotopy group of the generic fiber is trivial.*

Bibliography

[AIK13] F. Andreatta, A. Iovita, M. Kim, *A p-adic non-abelian criterion for good reduction of curves*, (2013), available at www.mat.unimi.it/users/andreat/research.html

[AM77] M. Artin, B. Mazur, *Formal groups arising from algebraic varieties*, Ann. Scient. École Norm. Sup. 4e série, t. 10 (1977), 87-132.

[Be97i] P. Berthelot, *Dualité de Poincaré et formule de Künneth en cohomologie rigide*, C. R. Acad. Sci. Paris, **325**, Série I (1997), pp. 493-498.

[Be97ii] P. Berthelot, *Finitude et pureté cohomologique en cohomologie rigide*. Invent. Math. **128** (1997), 329-377.

[BO78] P. Berthelot, A. Ogus, *Notes on Crystalline Cohomology*, Princeton University Press and University of Tokyo Press, 1978.

[BGR] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean Analysis: a systematic approach to rigid analytic geometry*. Grundlehren der Mathematischen Wissenschaften, Vol. 261. Springer-Verlag (1984).

[BPV] W. Barth, C. Peters, A. Ven, A. *Compact Complex Surfaces* Springer-Verlag (1984).

[Br00] C. Breuil, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489-549.

[Ch99] B. Chiarellotto, *Rigid Cohomology and Invariant Cycles for a Semistable Log Scheme*. Duke Mathematical Journal. **97** (1999), no.1, 155-169.

[Cl77] C.H. Clemens, *Degenerations of Kähler manifolds*. Duke Math J. **44** (1977), pp. 215-290.

[CD] F. R. Cossec, I. V. Dolgachev, *Enriques Surfaces I*, Progress in Math. vol. 76, Birkhauser (1989).

[Cr98] R. Crew, *Finiteness Theorems for the cohomology of an overconvergent isocrystal on a curve.* Ann. Sci. Éc. Norm. Supér., **31**, (1998), 717-763.

[CT03] B. Chiarellotto, N. Tsuzuki, *Cohomological descent of rigid cohomology for étale coverings*, Rend. Semin. Mat. Univ. Padova, **109**, (2003), 63-215.

[CT12] B. Chiarellotto, N. Tsuzuki, *Clemens-Schmid exact sequence in characteristic p* arXiv:1111.0779 (2012)

[CI99] R. Coleman, A. Iovita, *The Frobenius and monodromy operators for curves and abelian varieties*, Duke Mathematical Journal **97** (1999), no. 1, 171-215.

[De81] P. Deligne, *Relèvement des surfaces $K3$ en caractéristique nulle.* Surfaces algébriques, Séminaire de Géométrie algébrique d'Orsay 1976-1978. Lecture Notes in Math. 868, Springer-Verlag. New York (1981), pp. 58-79.

[El73] R. Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, Annales scientifiques de l'ENS, 4ème série, t. 6 (1973), pp. 553-604.

[FV] J. Fresnel, M. Van der Put, *Rigid Analytic Geometry and its Applications.* Progress in Mathematics, Vol. 218. Birkhäuser Boston Inc. (2004).

[HK94] O. Hyodo, K. Kato, *Semi-stable Reduction and Crystalline Cohomology with Logarithmic Poles*, Période p-adiques (Bures sur Yvette 1988), Astérisque, **223** (1994), 221-268.

[It05] T. Ito, *Weight-monodromy conjecture over equal characteristic local fields*, American Journal of Mathematics, vol. 127, no. 3, (2005), pp. 647-658.

[Ka96] F. Kato, *Log smooth deformation theory.* Tohoku Math. Jour. **48** (1996), 317-354.

[Ka89] K. Kato, *Logarithmic structures of Fontaine-Illusie.* Algebraic analysis, geometry and number theory, the Johns Hopkins Univ. Press (1989), pp. 191-224.

[Kw93] Y. Kawamata, *Semistable Minimal Models of Threefolds in Positive or Mixed Characteristic.* arXiv:alg-geom/9303001(1993).

[Kw98] Y. Kawamata, *Index 1 covers of log terminal surface singularities.* arXiv:math/9802044 (1998).

[Ku77] V. Kulikov, *Degenerations of* $K3$ *surfaces and Enriques' surfaces.* Math. USSR Izv. **11** (1977), 957-989.

[La73] A. Landman, *On the Picard-Lefschetz transformations.* Trans. AMS **181** (1973), pp. 89-126.

[Lz14] C. Lazda, *Rigid rational homotopy types.* arXiv:1306.6446 (2014).

[Li13] C. Liedtke, *Algebraic surfaces in positive characteristic* in Birational Geometry, Rational Curves and Arithmetic. F. Bogomolov, B. Hasset, Y. Tschnkel Eds., Springer 2013. 229-292.

[Liu] Q. Liu, *Algebraic Geometry and Arithmetic Curves.* Oxford Graduate Texts in Mathematics **6**. Oxford University Press 2002.

[LS] B. Le Stum, *Rigid Cohomology.* Cambridge Tracts in Mathematics **172**. Cambridge University Press 2007.

[LM14] C. Liedtke, Y. Matsumoro, *Good Reduction of* $K3$ *Surfaces*, arXiv:1411.4797 (2014).

[LP80] E. Looijenga, C. Peters, *Torelli theorems for Kähler K3 surfaces.* Compositio Mathematica, **42** (1980), 145?186.

[Ma14] Y. Matsumoto, *Good reduction criterion for* $K3$ *surfaces*, arXiv:1401.1261 (2014).

[Mau12] D. Maulik, *Supersingular K3 surfaces for large primes,* arXiv:1203.2889 (2012).

[Mo81] D.R. Morrison, *Semistable Degenerations of Enriques and Hyperelliptic Surfaces*, Duke Math. J., Vol. 48, No.1 (1981), 197-249.

[Mo84] D.R. Morrison, *The Clemens-Schmid exact sequence and applications,* Topics in Transcendental Algebraic Geometry, Annals of Mathematics Studies, vol. 106. Princeton University Press, Princeton (1984), pp. 101-119.

[Mk93] A. Mokrane, *La suite spectrale des poids en cohomologie de Hyodo-Kato.* Duke Math. J. **72** (1993), 301-337.

[Na00] Y. Nakkajima, *Liftings of Simple Normal Crossing Log $K3$ and Log Enriques Surfaces in Mixed Characteristics,* Algebraic Geometry, **9** (2000), 355-393.

[Ny83] N. O. Nygaard, *The Tate conjecture for ordinary $K3$ surfaces over finite fields*, Invent. Math. **74** (1983), 213-237.

[Oda95] T. Oda, *A note on ramification of the Galois representation on the fundamental group of an algebraic curve. II.* J. Number Theory **53** (1995), no. 2, 342-355.

[Pe14] J. R. Pérez Buendía, *A Crystalline Criterion for Good Reduction on Semi-stable $K3$-Surfaces over a p-Adic Field* (2014), available at http://spectrum.library.concordia.ca/978195/

[Po] D. Popescu, *General Néron Desingularization and Approximation.* Nagoya Math. J. **104** (1986), 85-115.

[PP81] U. Persson, H. Pinkham, *Degeneration of Surfaces with Trivial Canonical Bundle.* Annals of Mathematics, Second series, **113**, No.1, 45-66.

[Sc68] M. Schlessinger, *Functors of Artin Rings.* Trans. AMS **30** (1968), 208-222.

[Siu] Y. T. Siu, *Every K3 surface is Kähler*, Invent. Math. **73** (1983), no. 1, 139-150.

[ST68] J.P. Serre, J. Tate, *Good reduction of abelian varieties.* Ann. of Math. (2) **88** (1968), 492-517.

[Sh08] A. Shiho, *Relative Log Convergent Cohomology and Relative Rigid Cohomology I* arXiv:0707.1742 (2008).

[St76] J. Steenbrink, *Limits of Hodge Structures*, Inventiones mathematicae **31** (1976), no.3, 229-257.

[Sw95] R. Swan, Néron-Popescu desingularization. *Algebra and geometry* (Taipei 1995), Lect. Algebra Geom. 2 Int. Press, Cambridge, MA (1998), pp. 135-192.

[Ts99] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case.* Invent. Math. 137 (1999), **1**, 233-411.